Lec. No. (10)

Indeterminate Forms and L'Hopital's Rule

$$\lim_{x \to a} F(x) = f(a)$$

Determinate form

[Constant, ∞, $-\infty$]

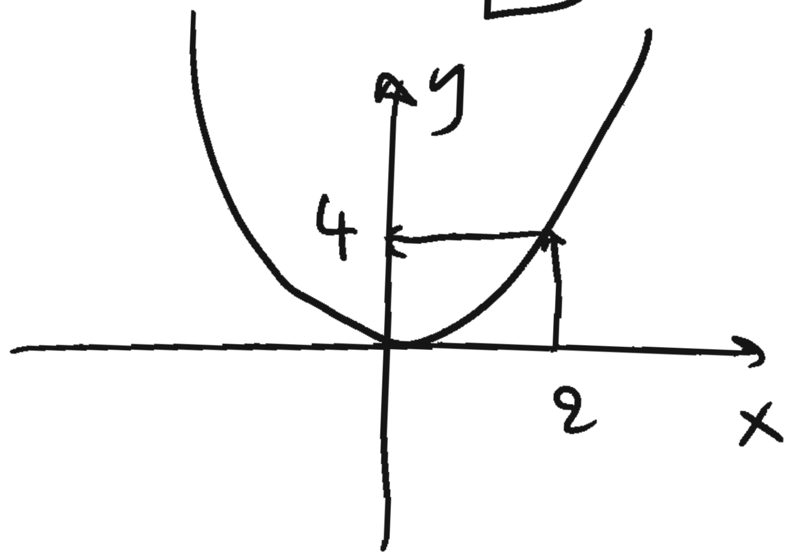
End

Indeterminate Form

[$\frac{0}{0}$, $\frac{\infty}{\infty}$, $0 * \infty$, $\infty - \infty$, 1^{∞}]

Try the get the value of the limit

Say, $\lim_{x \to 2} x^2 = (2)^2 = \boxed{4}$



$\lim_{x \to 1} \frac{x^2 - 1}{x - 1}$ $\quad \frac{0}{0}$

$= \lim_{x \to 1} \frac{\cancel{(x-1)}(x+1)}{\cancel{(x-1)}} = \boxed{2}$

L'Hôpital's Rule

Suppose that f(x) and g(x) are differentiable on the interval (a, b), except at the point $c \in (a,b)$ and that $g'(x) \neq 0$ on (a, b) except at c. and

$\lim_{x \to c} \frac{f(x)}{g(x)}$ has the indeterminate form $\frac{0}{0}$ or $\frac{\infty}{\infty}$

and that $\lim_{x \to c} \frac{f'(x)}{g'(x)} = L$ (or $\pm\infty$)

L ← constant

then, $$\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)}$$

Ex. $\lim\limits_{x\to 0} \dfrac{1-\cos x}{\sin x}$ $\quad \dfrac{0}{0}$

$= \lim\limits_{x\to 0} \dfrac{\sin x}{\cos x} = \dfrac{0}{1} = \boxed{0}$

Ex. $\lim\limits_{x\to\infty} \dfrac{e^x}{x}$ $\quad \dfrac{\infty}{\infty}$

$= \lim\limits_{x\to\infty} \dfrac{e^x}{1} = \dfrac{\infty}{1} = \boxed{\infty}$

Ex. $\lim\limits_{x\to\infty} \dfrac{x^2}{e^x}$ $\quad \dfrac{\infty}{\infty}$

$= \lim\limits_{x\to\infty} \dfrac{2x}{e^x}$ $\quad \dfrac{\infty}{\infty}$

$= \lim\limits_{x\to\infty} \dfrac{2}{e^x} = \dfrac{2}{\infty} = \boxed{0}$

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No.(1) $\lim_{x \to -2} \frac{x+2}{x^2-4}$ $\quad \frac{0}{0}$

$$= \lim_{x \to -2} \frac{1}{2x} = \boxed{-\frac{1}{4}}$$

No.3 $\lim_{x \to \infty} \frac{3x^2+2}{x^2-4}$ $\quad \frac{\infty}{\infty}$

$$= \lim_{x \to \infty} \frac{6x}{2x} = \boxed{3}$$

No.5 $\lim_{t \to 0} \frac{e^{2t}-1}{t}$ $\quad \frac{0}{0}$

$$= \lim_{t \to 0} \frac{2e^{2t}}{1} = \frac{2}{1} = \boxed{2}$$

No.7 $\lim\limits_{t\to 0} \dfrac{\tan^{-1} t}{\sin t}$ $\dfrac{0}{0}$

$$= \lim_{t\to 0} \frac{\frac{1}{1+t^2}}{\cos t} = \frac{1}{1} = \boxed{1}$$

No 9 $\lim\limits_{x\to \pi} \dfrac{\sin 2x}{\sin x}$ $\dfrac{0}{0}$

$$= \lim_{x\to \pi} \frac{2\cos 2x}{\cos x} = \frac{2}{-1} = \boxed{-2}$$

OR $\lim = \lim\limits_{x\to \pi} \dfrac{2\cancel{\sin x}\cos x}{\cancel{\sin x}} = \boxed{-2}$

No. 11 $\lim_{x \to 0} \frac{\sin x - x}{x^3}$ $\quad \frac{0}{0}$

$$= \lim_{x \to 0} \frac{\cos x - 1}{3x^2} \quad \frac{0}{0}$$

$$= \lim_{x \to 0} \frac{-\sin x}{6x} \quad \frac{0}{0}$$

$$= \lim_{x \to 0} \frac{-\cos x}{6} = \boxed{-\frac{1}{6}}$$

No. 13 $\lim_{t \to 1} \frac{\sqrt{t} - 1}{t - 1}$ $\quad \frac{0}{0}$

$$= \lim_{t \to 1} \frac{\frac{1}{2\sqrt{t}}}{1} = \frac{\frac{1}{2}}{1} = \boxed{\frac{1}{2}}$$

No.15 $\lim\limits_{x\to\infty} \frac{x^3}{e^x}$ $\quad \frac{\infty}{\infty}$

$$= \lim_{x\to\infty} \frac{3x^2}{e^x} \quad \frac{\infty}{\infty}$$

$$= \lim_{x\to\infty} \frac{6x}{e^x} \quad \frac{\infty}{\infty}$$

$$= \lim_{x\to\infty} \frac{6}{e^x} = \frac{6}{\infty} = \boxed{0}$$

No.17 $\lim\limits_{x\to 0} \frac{x\cos x - \sin x}{x\sin^2 x}$ $\quad \frac{0}{0}$

$$= \lim_{x\to 0} \frac{-x\sin x + \cos x - \cos x}{x \cdot 2\sin x\cos x + \sin^2 x \cdot 1}$$

$$= \lim_{x \to 0} \frac{-x \sin x}{x \sin 2x + \sin^2 x} \quad \frac{0}{0}$$

$$= \lim_{x \to 0} \frac{-x \cos x - \sin x}{2x \cos 2x + \sin 2x \cdot 1 + 2 \sin x \cos x}$$

$$= \lim_{x \to 0} \frac{-x \cos x - \sin x}{2x \cos 2x + 2 \sin 2x} \quad \frac{0}{0}$$

$$= \lim_{x \to 0} \frac{x \sin x - \cos x \cdot 1 - \cos x}{-4x \sin 2x + 2 \cos 2x + 4 \cos 2x}$$

$$= \frac{0 - 1 - 1}{0 + 2 + 4} = \frac{-2}{6} = \boxed{-\frac{1}{3}}$$

No. 19 $\lim_{x \to 0} \left(\frac{x+1}{x} - \frac{2}{\sin 2x} \right)$

$\infty - \infty$

$$= \lim_{x \to 0} \frac{(x+1)\sin 2x - 2x}{x \sin 2x} \qquad \frac{0}{0}$$

$$= \lim_{x \to 0} \frac{2(x+1)\cos 2x + \sin 2x - 2}{2x\cos 2x + \sin 2x} \qquad \frac{0}{0}$$

$$= \lim_{x \to 0} \frac{-4(x+1)\sin 2x + 2\cos 2x + 2\cos 2x}{-4x\sin 2x + 2\cos 2x + 2\cos 2x}$$

$$= \frac{0+2+2}{0+2+2} = \frac{4}{4} = \boxed{1}$$

No 21 $\lim_{x\to\infty} \frac{\ln x}{x^2}$ $\quad \frac{\infty}{\infty}$

$$= \lim_{x\to\infty} \frac{\frac{1}{x}}{2x} = \lim_{x\to\infty} \frac{1}{2x^2}$$

$$= \frac{1}{\infty} = \boxed{0}$$

No.23 $\lim_{t\to\infty} te^{-t}$ $\quad \infty * 0$

$$= \lim_{t\to\infty} \frac{t}{e^t} \quad \frac{\infty}{\infty}$$

$$= \lim_{t\to\infty} \frac{1}{e^t} = \frac{1}{\infty} = \boxed{0}$$

No 25 $\lim_{t\to 1} \dfrac{\ln(\ln t)}{\ln t}$ $\frac{0}{0}$

$$= \lim_{t\to 1} \frac{\frac{1}{\ln t}\cdot\frac{1}{t}}{\cancel{\frac{1}{t}}} = \frac{1}{0} = \boxed{\infty}$$

No. 27 $\lim_{x\to 0} \dfrac{\sin(\sinh x)}{\sinh(\sin x)}$ $\frac{0}{0}$

$$= \lim_{x\to 0} \frac{\cos(\sinh x)\cdot\cosh x}{\cosh(\sin x)\cdot\cos x} = \frac{1*1}{1*1} = \boxed{1}$$

No. 29 $\lim_{x \to 0} \frac{\ln x}{\cot x}$ $\quad \frac{\infty}{\infty}$

$$= \lim_{x \to 0} \frac{\frac{1}{x}}{-\csc^2 x} = -\lim_{x \to 0} \frac{\sin^2 x}{x} \quad \frac{0}{0}$$

$$= -\lim_{x \to 0} \frac{2\sin x \cos x}{1} = \frac{0}{1} = \boxed{0}$$

No 31 $\lim_{x \to \infty} \left(\sqrt{x^2-1} - x\right)$ $\quad \infty - \infty$

$$= \lim_{x \to \infty} \left(\sqrt{x^2-1} - x\right) \frac{\left(\sqrt{x^2-1} + x\right)}{\left(\sqrt{x^2-1} + x\right)}$$

$$= \lim_{x \to \infty} \frac{(x^2-1) - x^2}{\sqrt{x^2-1} + x} = \lim_{x \to \infty} \frac{-1}{\sqrt{x^2-1} + x}$$

$$= \frac{-1}{\infty} = \boxed{0}$$

No. 33 $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x}$ 1^{∞}

$$= \boxed{e}$$

Note

$$\lim_{\square\to\infty}\left(1+\frac{1}{\square}\right)^{\square} = e$$

$$\lim_{\square\to\infty}\left(1+\frac{a}{\square}\right)^{b\square} = e^{ab}$$

$$\lim_{\square\to 0}\left(1+\square\right)^{\frac{1}{\square}} = e$$

Sheet No. (9)

1) $\lim\limits_{x\to\pi/2} \dfrac{2\cos x}{2x-\pi}$ $\quad \dfrac{0}{0}$

$= \lim\limits_{x\to\frac{\pi}{2}} \dfrac{-2\sin x}{2} = \dfrac{-2\times 1}{2} = \boxed{-1}$

(2) $\lim\limits_{x\to o} \dfrac{1-\cosh x}{x^2}$ $\quad \dfrac{0}{0}$

$= \lim\limits_{x\to 0} \dfrac{-\sinh x}{2x}$ $\quad \dfrac{0}{0}$

$= \lim\limits_{x\to 0} -\dfrac{\cosh x}{2} = \boxed{-\dfrac{1}{2}}$

(3) $\lim\limits_{x\to o} \dfrac{\sqrt{1+\sin x}-\sqrt{1-\sin x}}{x}$ $\quad \dfrac{0}{0}$

$= \lim\limits_{x\to 0} \dfrac{\frac{1}{2\sqrt{1+\sin x}}\cdot\cos x - \frac{1}{2\sqrt{1-\sin x}}\cdot(-\cos x)}{1}$

$= \dfrac{\frac{1}{2\sqrt{1+0}}\cdot(1) + \frac{1}{2\sqrt{1-0}}\cdot(1)}{1} = \dfrac{\frac{1}{2}+\frac{1}{2}}{1}$

$= \boxed{1}$

(4) $\lim\limits_{x\to 1}(1-x)\tan\frac{\pi x}{2}$ $\qquad 0 * \infty$

$$= \lim_{x\to 1} \frac{(1-x)\sin\frac{\pi x}{2}}{\cos\frac{\pi x}{2}} \qquad \frac{0}{0}$$

$$= \lim_{x\to 1} \frac{(1-x)\cdot\cos\frac{\pi x}{2}\cdot\frac{\pi}{2} + \sin\frac{\pi}{2}x\cdot(-1)}{-\sin\frac{\pi}{2}x\cdot\frac{\pi}{2}}$$

$$= \frac{0-1}{-\frac{\pi}{2}} = \boxed{\frac{2}{\pi}}$$

5) $\lim\limits_{x\to 0}\frac{x\cos x+\tan 2x}{x\sec x+\sin 4x}$ $\qquad \frac{0}{0}$

$\sec 0 = \frac{1}{\cos 0} = \frac{1}{1} = 1$

$$= \lim_{x\to 0} \frac{-x\sin x+\cos x\cdot 1+2\sec^2 2x}{x\sec x\tan x+\sec x\cdot 1+4\cos 4x}$$

$$= \frac{0+1+2}{0+1+4} = \boxed{\frac{3}{5}}$$

(6) $\lim_{x \to o}(\operatorname{cosec}\phi - \cot\phi)$ $\quad (\infty - \infty)$

$$= \lim_{x \to 0}\left[\frac{1}{\sin\phi} - \frac{\cos\phi}{\sin\phi}\right]$$

$$= \lim_{x \to 0} \frac{1-\cos\phi}{\sin\phi} \quad \frac{0}{0}$$

$$= \lim_{x \to 0} \frac{\sin\phi}{\cos\phi} = \frac{0}{1} = \boxed{0}$$

$$\operatorname{Cosec} 0 = \frac{1}{\sin 0} = \frac{1}{0} = \infty$$

$$\operatorname{Cot} 0 = \frac{1}{\tan 0} = \infty$$

(7) $\lim\limits_{x\to\infty}\left(\dfrac{x+3}{x-1}\right)^{x}$ $\qquad \left(\dfrac{1}{1}\right)^{\infty} \qquad 1^{\infty}$

$$= \lim_{x\to\infty}\left(\frac{(x-1)+4}{(x-1)}\right)^{x}$$

$$= \lim_{x\to\infty}\left[1+\frac{4}{x-1}\right]^{x}$$

$$= \lim_{\substack{x\to\infty \\ x-1\to\infty}}\left[1+\frac{4}{x-1}\right]^{x-1} \lim_{x\to\infty}\frac{x}{x-1}$$

$$= e^{4\times 1}$$

$$= \boxed{e^{4}}$$

(8) $\lim_{x\to o}(\cos x)^{1/x^2}$

1^{∞}

let $y = \lim_{x\to 0} (\cos x)^{\frac{1}{x^2}}$

$\ln y = \lim_{x\to 0} \frac{\ln\cos x}{x^2} \quad \frac{0}{0}$

$= \lim_{x\to 0} \frac{\frac{1}{\cos x}\cdot -\sin x}{2x}$

$= \lim_{x\to 0} \frac{-\tan x}{2x} \quad \frac{0}{0}$

$= \lim_{x\to 0} \frac{-\sec^2 x}{2} = -\frac{1}{2}$

$y = e^{-\frac{1}{2}} \quad \therefore \lim_{x\to 0} (\cos x)^{\frac{1}{x^2}} = e^{-\frac{1}{2}} = \boxed{\frac{1}{\sqrt{e}}}$

$\underline{\text{Ex}}$ $\lim\limits_{x\to 0}(1+\sin x)^{\frac{1}{x}}$ $\quad 1^{\infty}$

Let $y=\lim\limits_{x\to 0}(1+\sin x)^{\frac{1}{x}}$

$$\ln y=\lim_{x\to 0}\frac{\ln(1+\sin x)}{x} \qquad \frac{0}{0}$$

$$=\lim_{x\to 0}\frac{\frac{1}{1+\sin x}\cdot\cos x}{1}=\frac{1}{1}=1$$

$\ln y=1 \qquad y=e^{1}$

$\therefore \lim\limits_{x\to 0}(1+\sin x)^{\frac{1}{x}}=1$

$\underline{\text{OR}}$ $\lim=\lim\limits_{\substack{x\to 0\\ \sin x\to 0}}(1+\sin x)^{\frac{1}{\sin x}\lim\limits_{x\to 0}\frac{\sin x}{x}}=e^{1}=e$