Lec. No. (3)

Inverse Trigonometric functions

You know

① $\sec x = \dfrac{1}{\cos x} = (\cos x)^{-1}$

$(\cos x)^{-1} \neq \cos^{-1} x$

Not a number
It's a symbol

sec x

Inverse function for cos x

② If $y = \sin x$

$\therefore x = \sin^{-1} y$

If $y = \tan^{-1} x$

$\therefore x = \tan y$

(3) $\tan \tan^{-1} 1 = \tan \frac{\pi}{4} = 1$

$\tan^{-1} \tan \frac{\pi}{4} = \tan^{-1} 1 = \frac{\pi}{4}$

$$\boxed{\begin{array}{l} \sin^{-1} \sin \square = \square \\ \sec \sec^{-1} \square = \square \end{array}}$$

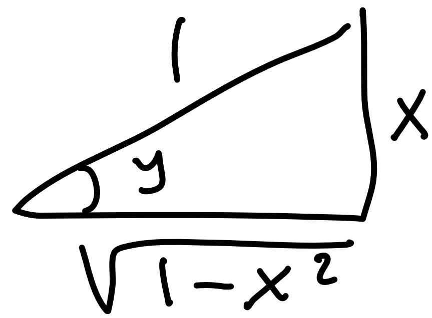
<u>Derivatives of Inverse Trigonometric Functions</u>

y	$y' = \frac{dy}{dx}$
$y = \sin^{-1} u$	$y' = \frac{1}{\sqrt{1-u^2}} . u'$
$y = \cos^{-1} u$	$y' = \frac{-1}{\sqrt{1-u^2}} . u'$
$y = \tan^{-1} u$	$y' = \frac{1}{1+u^2} . u'$
$y = \cot^{-1} u$	$y' = \frac{-1}{1+u^2} . u'$
$y = \sec^{-1} u$	$y' = \frac{1}{u\sqrt{u^2-1}} . u'$
$y = \operatorname{cosec}^{-1} u$	$y' = \frac{-1}{u\sqrt{u^2-1}} . u'$

$* \quad y = \sin^{-1} x$

$$x = \sin y \Rightarrow \frac{dx}{dy} = \cos y$$

$$\therefore \frac{dx}{dy} = \sqrt{1-x^2}$$

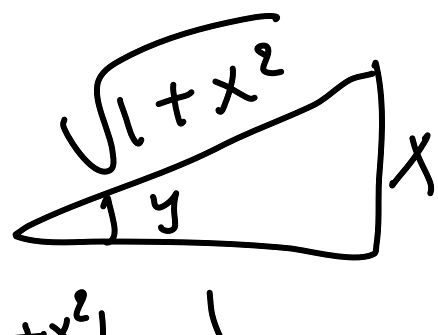


$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} = \frac{1}{\sqrt{1-x^2}}$$

$* \quad y = \tan^{-1} x$

$$x = \tan y \Rightarrow \frac{dx}{dy} = \sec^2 y$$

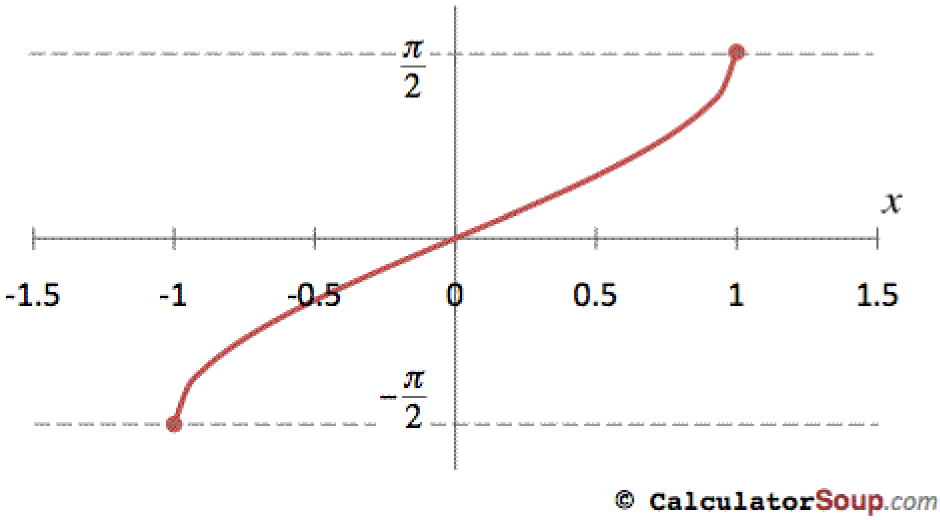
$$\cos y = \frac{1}{\sqrt{1+x^2}}$$



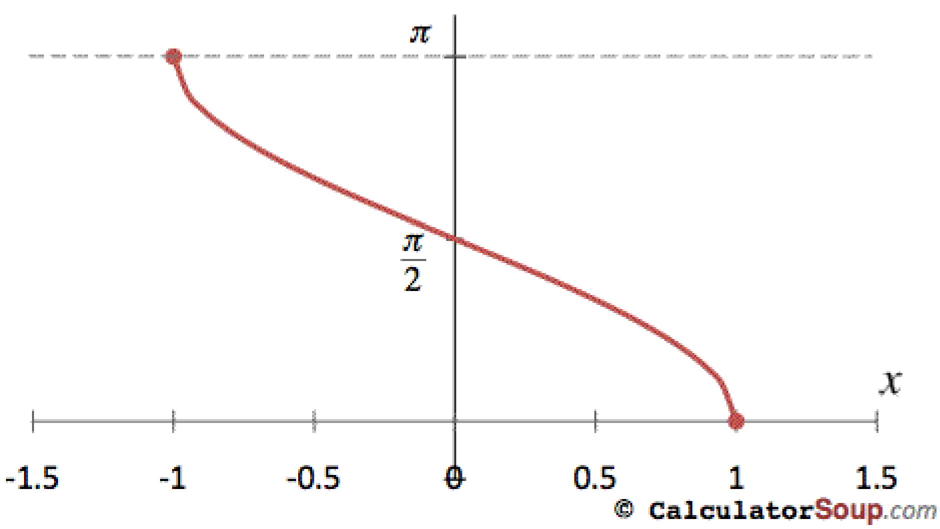
$$\cos^2 y = \frac{1}{1+x^2} \Rightarrow \sec^2 y = (1+x^2)$$

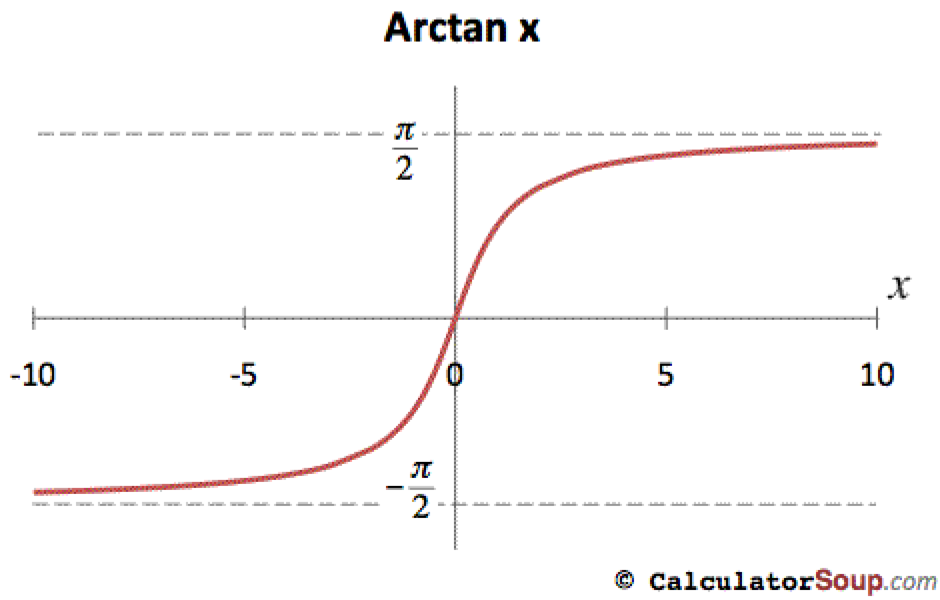
$$\therefore \frac{dy}{dx} = \frac{1}{1+x^2}$$

Arcsin x



Arccos x




Arctan x
$\frac{\pi}{2}$
x
-10
-5
0
5
10
$-\frac{\pi}{2}$
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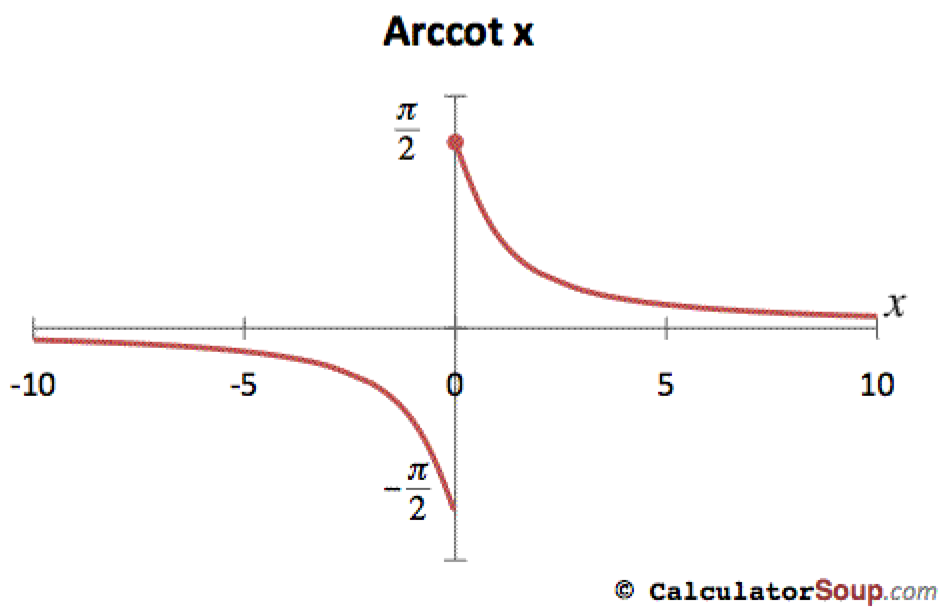
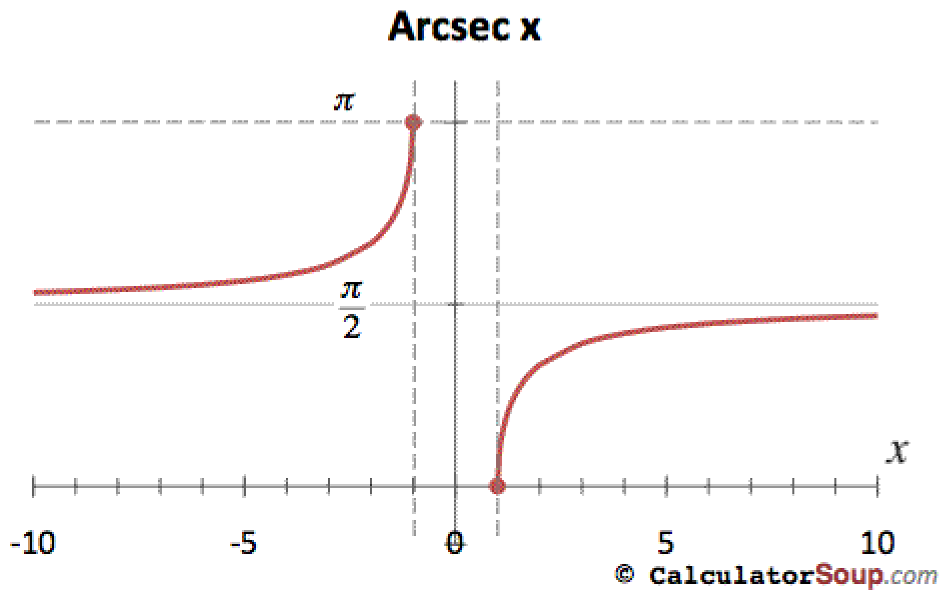

Arccot x
$\frac{\pi}{2}$
x
-10
-5
0
5
10
$-\frac{\pi}{2}$
© CalculatorSoup.com


Arcsec x
π
$\frac{\pi}{2}$
x
-10
-5
0
5
10
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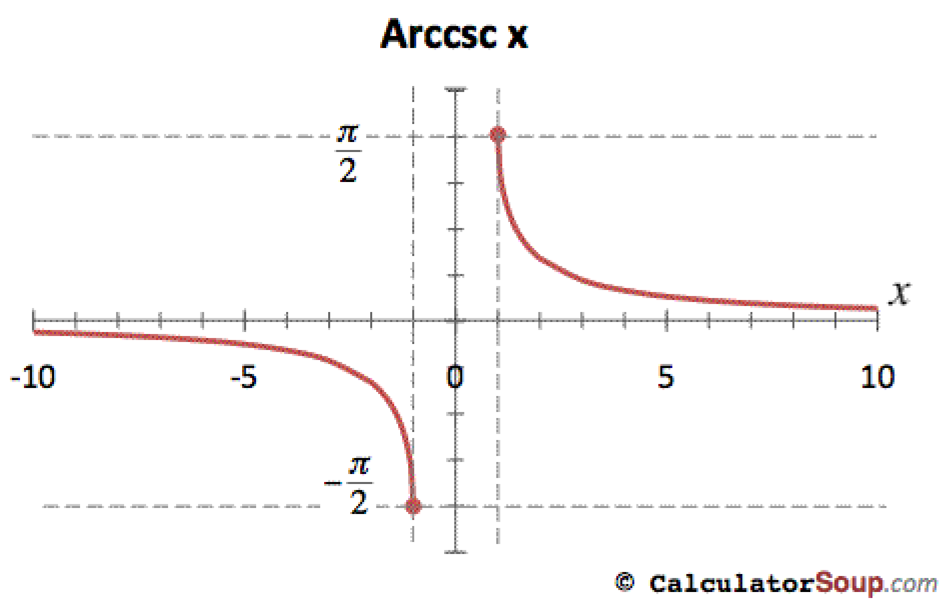
Arccsc x
$\frac{\pi}{2}$
x
-10
-5
0
5
10
$-\frac{\pi}{2}$
© CalculatorSoup.com

Ex. $y = \sin^{-1} x^2 \Rightarrow y' = \dfrac{1}{\sqrt{1-x^4}} \cdot 2x$

Ex. $y = \cos^{-1} 3x \Rightarrow y' = \dfrac{-1}{\sqrt{1-9x^2}} \cdot 3$

Ex. $y = \tan^{-1} \sqrt{x} \Rightarrow y' = \dfrac{1}{1+x} \cdot \dfrac{1}{2\sqrt{x}}$

Ex. $y = \cot^{-1} x^2 \Rightarrow y' = \dfrac{-1}{1+x^4} \cdot 2x$

Ex. $y = \sec^{-1} \sqrt{x} \Rightarrow y' = \dfrac{1}{\sqrt{x}\sqrt{x-1}} \cdot \dfrac{1}{2\sqrt{x}}$

Ex. $y = \csc^{-1} x^3 \Rightarrow y' = \dfrac{-1}{(x^3)\sqrt{x^6-1}} \cdot 3x^2$

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No. 29 (a) $f(x) = \sin^{-1}(x^3+1)$

$$f' = \frac{1}{\sqrt{1-(x^3+1)^2}} \cdot 3x^2$$

(b) $f(x) = \sin^{-1}(\sqrt{x})$

$$f' = \frac{1}{\sqrt{1-x}} \cdot \frac{1}{2\sqrt{x}}$$

No. 31 (a) $f(x) = \tan^{-1}\sqrt{x}$

$$f' = \frac{1}{1+x} \cdot \frac{1}{2\sqrt{x}}$$

(b) $f(x) = \tan^{-1}\left(\frac{1}{x}\right)$

$$f' = \frac{1}{1+\frac{1}{x^2}} \cdot \frac{-1}{x^2} = \frac{-1}{x^2+1}$$

sheet (3)

(a) (1) $y = \cos^{-1}\sqrt{x} \Rightarrow y' = \dfrac{-1}{\sqrt{1-x}} \cdot \dfrac{1}{2\sqrt{x}}$

(2) $y = (x^2+4)\csc^{-1} 2x$

$$y' = (x^2+4) \cdot \frac{-1}{(2x)\sqrt{4x^2-1}} \cdot 2 + (\csc^{-1} 2x)(2x)$$

(3) $y = x^3\left(1 - \sec^{-1}\sqrt{x}\right)$

$$y' = x^3\left[\frac{-1}{\sqrt{x}\sqrt{x-1}} \cdot \frac{1}{2\sqrt{x}}\right] + (1-\sec^{-1}\sqrt{x})(3x^2)$$

(4) $y = x^3 \sin^{-1}\sqrt{x} - 2\cot^{-1} x^2$

$$y' = x^3 \cdot \frac{1}{\sqrt{1-x}} \cdot \frac{1}{2\sqrt{x}} + \sin^{-1}\sqrt{x} \cdot 3x^2$$

$$-2 \cdot \frac{-1}{1+x^4} \cdot 2x$$

⑤ $y = \tan^{-1}\left(\frac{x-1}{x+1}\right)$

$$y' = \frac{1}{1+\left(\frac{x-1}{x+1}\right)^2} \cdot \frac{(x+1)\cdot 1-(x-1)\cdot 1}{(x+1)^2}$$

$$y' = \frac{2}{(x+1)^2+(x-1)^2}$$

⑥ $y = \frac{1-\sin^{-1}x}{\cos^{-1}x}$

$$y' = \frac{(\cos^{-1}x)\left[-\frac{1}{\sqrt{1-x^2}}\right]-(1-\sin^{-1}x)\left[\frac{-1}{\sqrt{1-x^2}}\cdot 1\right]}{(\cos^{-1}x)^2}$$

(b) If $y = \tan(\cos^{-1} x)$

show $y' = \dfrac{-(y^2+1)}{\sqrt{1-x^2}}$

Proof

$$y' = \sec^2(\cos^{-1} x) \frac{-1}{\sqrt{1-x^2}}$$

Recall $\sec^2\theta = 1 + \tan^2\theta$

$$\therefore \sec^2(\cos^{-1} x) = 1 + \tan^2(\cos^{-1} x)$$
$$= 1 + y^2$$

$$\therefore y' = \frac{-(1+y^2)}{\sqrt{1-x^2}}$$

O.K

(f) Prove that $\frac{d}{dx}\left(\tan^{-1}\frac{x-1}{x+1}\right)=\frac{d}{dx}(\tan^{-1}x)$

Proof

R.H.S : $\frac{d}{dx}(\tan^{-1}x) = \frac{1}{1+x^2}$

L.H.S : $\frac{d}{dx}\left(\tan^{-1}\frac{x-1}{x+1}\right)$

$$= \frac{1}{1+\left(\frac{x-1}{x+1}\right)^2} \cdot \frac{(x+1)\cdot 1-(x-1)\cdot 1}{(x+1)^2}$$

$$= \frac{2}{(x+1)^2+(x-1)^2}$$

$$= \frac{2}{x^2+\cancel{2x}+1+x^2-\cancel{2x}+1}$$

$$= \frac{2}{2x^2+2} = \frac{1}{1+x^2} = \text{R.H.S}$$

o.k

Ex.

If $y = \sec(\tan^{-1}x) + \operatorname{cosec}(\cot^{-1}x)$

show that $y' = \dfrac{xy}{1+x^2}$

Proof

$$y' = \sec(\tan^{-1}x)\cancel{\tan(\tan^{-1}x)}^{\,x} \cdot \frac{1}{1+x^2}$$

$$+$$

$$-\operatorname{cosec}(\cot^{-1}x)\cancel{\cot(\cot^{-1}x)}^{\,x} \cdot \frac{-1}{1+x^2}$$

$$y' = \frac{x}{1+x^2}\sec(\tan^{-1}x) + \frac{x}{1+x^2}\operatorname{cosec}(\cot^{-1}x)$$

$$= \frac{x}{1+x^2}\underbrace{\left[\sec(\tan^{-1}x) + \operatorname{cosec}(\cot^{-1}x)\right]}_{y}$$

$$y' = \frac{xy}{1+x^2} \qquad \underline{\text{O.K}}$$