Math 2

Lecture 5

Basic integration techniques

Integrations of powers of trigonometric functions

Main objectives

1. *Integrating products of sine and cosine functions with various powers.*
2. *Integrating products of tangent and secant functions with various powers.*
3. *Integrating products of sine and cosine functions with different arguments.*

Integrals involving powers of Trigonometric Functions

- Evaluating an integral whose integrand contains powers of one or more trigonometric functions often involves making a clever substitution.

We first consider integrals of the form

$$\int \sin^m x \cos^n x \; dx,$$

where m and n are positive integers

There are two cases

Case 1: m or n is an odd positive integer

If m is odd, first isolate one factor of $\sin x$ (you will need this for du)

Then, replace any actors of $\sin^2 x$ with $1-\cos^2 x$ and make the substitution $u = \cos x$.

Likewise If n is odd, first isolate one factor of $\cos x$ (you will need this for du)

Then, replace any actors of $\cos^2 x$ with $1-\sin^2 x$ and make the substitution $u = \sin x$.

Example 1(Case 1)

Evaluate $\int \cos^4 x \sin x \, dx$

Solution

Let $u = \cos x$, so that $du = -\sin x \, dx$, *then*

$$\int \cos^4 x \sin x \, dx = -\int \cos^4 x (-\sin x) \, dx = -\int u^4 \, du$$

$$= -\frac{u^5}{5} + C = -\frac{\cos^5 x}{5} + C$$

Example 2 (Case 1): An Integrand with an odd power of sine

Evaluate $\int \cos^4 x \sin^3 x \, dx.$

Solution

Let $u = \cos x$, *we have* $du = -\sin x$, *so that*

$$\int \cos^4 x \sin^3 x \, dx = \int \cos^4 x \sin^2 x \sin x \, dx = -\int \cos^4 x \sin^2 x (-\sin x) \, dx$$

$$= -\int \cos^4 x (1 - \cos^2 x)(-\sin x) \, dx = -\int u^4 (1 - u^2) \, du$$

$$= -\int (u^4 - u^6) \, du = -\left(\frac{u^5}{5} - \frac{u^7}{7}\right) + C$$

$$= -\frac{\cos^5 x}{5} + \frac{\cos^7 x}{7} + C$$

Example 2 (Case 1): An integrand with an odd power of cosine

Evaluate $\int \sqrt{\sin x}\cos^5 x\,dx$

Solution :

we rewrite the integral as

$$\int \sqrt{\sin x}\cos^5 x\,dx = \int \sqrt{\sin x}(\cos^4 x)\cos x\,dx = \int \sqrt{\sin x}\,(1-\sin^2 x)^2 \cos x\,dx$$

Substituting $u = \sin x$, so that $du = \cos x\,dx$, *we have*

$$\int \sqrt{\sin x}\cos^5 x\,dx = \int \sqrt{u}(1-u^2)^2\,du = \int u^{1/2}(1-2u^2+u^4)\,du$$

$$= \int (u^{1/2} - 2u^{5/2} + u^{9/2})\,du$$

$$= \frac{2}{3}u^{3/2} - \frac{4}{7}u^{7/2} + \frac{2}{11}u^{11/2} + C$$

$$= \frac{2}{3}\sin^{3/2} x - \frac{4}{7}\sin^{7/2} x + \frac{2}{11}\sin^{11/2} x + C$$

Case 2: m and n are both even positive integers

If both m and n are even in $\int \sin^m x \cos^n x \, dx$, we substitute

$$\sin^2 x = \frac{1 - \cos 2x}{2}, \qquad \cos^2 x = \frac{1 + \cos 2x}{2}$$

to reduce the integrand to one in lower powers of $\cos 2x$.

Example (Case 2) : An Integrand with an even power of sine

Evaluate $\int \cos^4 x \, dx$

Solution

using the relation $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$

$$\int \cos^4 x \, dx = \frac{1}{4}\int (1 + \cos 2x)^2 \, dx$$

$$= \frac{1}{4}\int (1 + 2\cos 2x + \cos^2 2x) \, dx$$

but $\cos^2 2x = \frac{1}{2}(1 + \cos 4x)$, *so*

$$\int \cos^4 x \, dx = \frac{1}{4}\int \left(1 + 2\cos 2x + \frac{1}{2}(1 + \cos 4x)\right) dx$$

$$= \frac{3}{8}x + \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x + C$$

Example 3 (Case 2):

Evaluate $\int \sin^2 x \cos^4 x\,dx$

Solution:

$$\int \sin^2 x \cos^4 x\,dx = \int \frac{1}{2}(1-\cos 2x).\frac{1}{4}(1+\cos 2x)^2\,dx$$

$$= \int \frac{1}{8}(1-\cos^2 2x)(1+\cos 2x)\,dx$$

$$= \frac{1}{8}\int (1+\cos 2x-\cos^2 2x-\cos^3 2x)\,dx$$

$$= \frac{1}{8}\int [1+\cos 2x-\frac{1}{2}(1+\cos 4x)-\cos^2 2x.\cos 2x]\,dx$$

$$= \frac{1}{8}\int [1+\cos 2x-\frac{1}{2}(1+\cos 4x)-(1-\sin^2 2x).\cos 2x]\,dx$$

$$= \frac{1}{8}\int [\frac{1}{2}-\frac{1}{2}\cos 4x+\sin^2 2x\cos 2x]\,dx$$

$$= \frac{1}{8}[\frac{1}{2}x-\frac{1}{8}\sin 4x+\frac{1}{6}\sin^3 2x]+C$$

Evaluating Integrals of the form $\int \tan^m x \sec^n x \, dx$

We have 3 cases:

1. If power of secant is even, save a factor of $\sec^2 x$ and use

$$\sec^2 x = 1 + \tan^2 x$$

to express the remaining factors in terms of tangent. Then substitute *u = tanx*.

2. If power of tangent is odd, save a factor of *secx tanx* and use

$$\tan^2 x = \sec^2 x - 1$$

to express the remaining factors in terms of the secant. Then substitute *u = secx*.

3. If m is even and n is odd we convert the product into a power secant function only.

Example 4 : Case 1

$$\int tan^6 x\, sec^4 x\, dx = \int tan^6 x\, sec^2 x\, \underline{sec^2 x}\,\underline{dx}$$

$$sec^2 x = 1 + tan^2 x$$

$$= \int tan^6 x\left(1 + tan^2 x\right)\underline{sec^2 x}\,\underline{dx}$$

$$= \int u^6\left(1 + u^2\right)du = \int (u^6 + u^8)du$$

$$= \frac{1}{7}u^7 + \frac{1}{9}u^9 + C = \frac{1}{7}tan^7 x + \frac{1}{9}tan^9 x + C$$

Example 5: (Case 2)

$$\int \tan^5 x \sec^7 x\, dx = \int \tan^4 x \sec^6 x . \underline{\tan x \sec x dx}$$

$$\tan^2 x = \sec^2 x - 1$$

$$= \int \left(\sec^2 x - 1\right)^2 \sec^6 x . \underline{\tan x \sec x dx}$$

$$u = \sec x \Rightarrow$$

$$du = \sec x \tan x dx$$

$$= \int u^6 \left(u^2 - 1\right)^2 du = \int u^6 \left(u^4 - 2u^2 + 1\right) du$$

$$= \int \left(u^{10} - 2u^8 + u^6\right) du = \frac{1}{11} u^{11} - \frac{2}{9} u^9 + \frac{1}{7} u^7 + C$$

$$= \frac{1}{11} \sec^{11} x - \frac{2}{9} \sec^9 x + \frac{1}{7} \sec^7 x + C$$

Example 6 (Case 3)

$$\int \tan^2 x \sec^3 x \, dx = \int (\sec^2 x - 1) \sec^3 x dx$$

$$= \int (\sec^5 x - \sec^3 x) dx$$

The integrals can be found using integration by parts repeatedly

Integrals of the $(a)\int \sin mx \cos nx dx,$

$(b)\int \sin mx \sin nx dx,$

$(c)\int \cos mx \cos nx dx$

Can be evaluated easily using the corresponding identity

$$(a) \sin mx \cos nx = \frac{1}{2}\left[\sin(m-n)x + \sin(m+n)x\right]$$

$$\text{(b)} \sin mx \sin nx = \frac{1}{2}\left[\cos(m-n)x - \cos(m+n)x\right]$$

$$\text{(c)} \cos mx \cos nx = \frac{1}{2}\left[\cos(m-n)x + \cos(m+n)x\right]$$

Example 7

$$(1)\ \int \sin 4x \cos 5x\, dx = \int \frac{1}{2}[\sin(-x) + \sin(9x)]dx$$

using identity (a)

$$= \int \frac{1}{2}[-\sin x + \sin 9x]dx = \frac{1}{2}\left(\cos x - \frac{1}{9}\cos 9x\right) + C$$

using identity (c)

$$(2)\ \int \cos 7x \cos 3x\, dx = \int \frac{1}{2}[\cos(4x) + \cos(10x)]dx$$

$$= \frac{1}{2}\left[\frac{1}{4}\sin(4x) + \frac{1}{10}\sin s(10x)\right] + C$$