Math 2

Lecture 6

Basic integration techniques

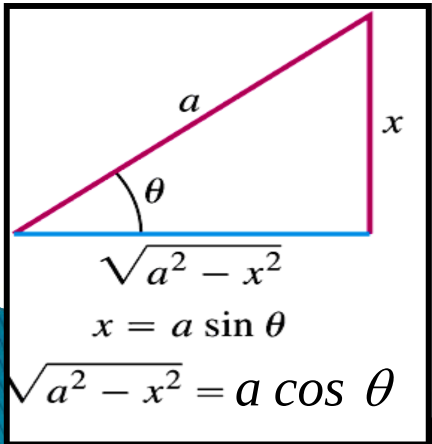
Trigonometric Substitution

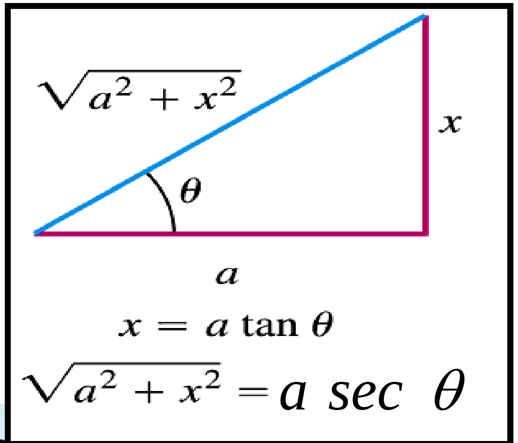
Objectives

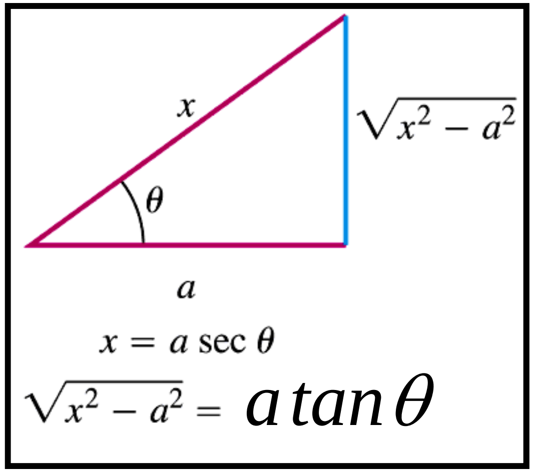
1. *Calculating integrals involving* $\sqrt{a^2 - x^2}$
2. *Calculating integrals involving* $\sqrt{a^2 + x^2}$
3. *Calculating integrals involving* $\sqrt{x^2 - a^2}$
4. *Study further extensions of the above cases*

Integration by Trigonometric Substitutions

Integrand contains:	Substitution	Identity used
$\sqrt{a^2 - x^2}$	$x = a\sin\theta, \quad -\pi/2 \le \theta \le \pi/2$	$1 - \sin^2\theta = \cos^2\theta$
$\sqrt{a^2 + x^2}$	$x = a\tan\theta, \quad -\pi/2 < \theta < \pi/2$	$1 + \tan^2\theta = \sec^2\theta$
$\sqrt{x^2 - a^2}$	$x = a\sec\theta, \quad 0 \le \theta < \pi/2$ or $\pi \le \theta < 3\pi/2$	$\sec^2\theta - 1 = \tan^2\theta$







Example 1: *Find the value of* $I = \int x^3 \sqrt{9 - x^2}\, dx$

Let $x = 3\sin\theta \Rightarrow dx = 3\cos\theta\, d\theta$

$\therefore \sqrt{9 - x^2} = \sqrt{9 - 9\sin^2\theta} = 3\sqrt{\cos^2\theta} = 3\cos\theta$

$\therefore I = \int x^3 \sqrt{9 - x^2}\, dx = \int 27\sin^3\theta . 3\cos\theta . 3\cos\theta\, d\theta = \int 243\cos^2\theta . \sin^3\theta\, d\theta$

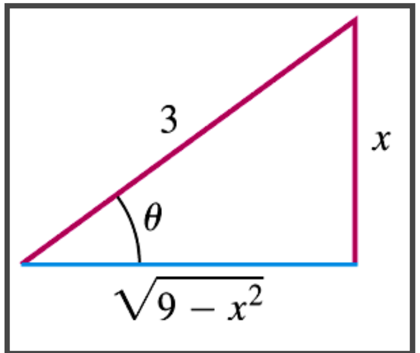
$= 243\int \cos^2\theta . (1 - \cos^2\theta)\sin\theta\, d\theta$

Let $u = \cos\theta \Rightarrow du = -\sin\theta\, d\theta$

$\therefore I = -243\int u^2(1 - u^2)\, du = -\frac{243}{3}u^3 + \frac{243}{5}u^5 + C$

$= -\frac{243}{3}\cos^3\theta + \frac{243}{5}\cos^5\theta + C$

but $\cos\theta = \frac{\sqrt{9 - x^2}}{3}$

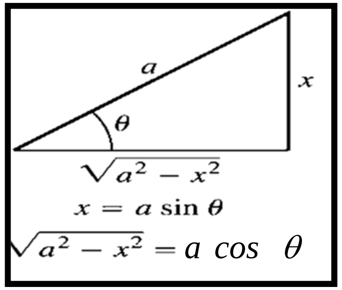


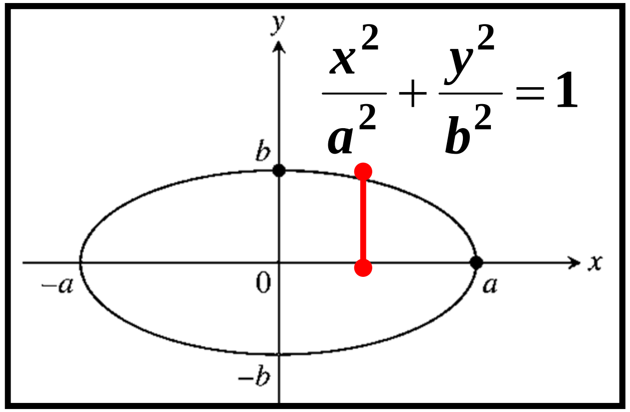
$\sqrt{9 - x^2} = 3cos\theta$

$\therefore I = -81\left(\frac{\sqrt{9 - x^2}}{3}\right)^3 + \frac{243}{5}\left(\frac{\sqrt{9 - x^2}}{3}\right)^5 + C$

Example 2: Find the area enclosed by the ellipse

$$A = 4\int_0^a \frac{b}{a}\sqrt{a^2 - x^2}\,dx$$





Let $x = a\sin\theta$

$$x = 0 \Rightarrow \sin\theta = 0 \Rightarrow \theta = 0$$

$$x = a \Rightarrow \sin\theta = 1 \Rightarrow \theta = \pi/2$$

$$\therefore A = 4\int_0^{\pi/2} \left(\frac{b}{a}\right) a\cos\theta . a\cos\theta\, d\theta = 4ab\int_0^{\pi/2} \cos^2\theta\, d\theta$$

$$= 4ab\int_0^{\pi/2} \frac{1}{2}\left(1 + \cos 2\theta\right) d\theta = 2ab\left(\theta + \frac{1}{2}\sin 2\theta\right)_0^{\pi/2} = \pi ab$$

Example 3: Find the value of $I = \int \frac{1}{x^2\sqrt{4+x^2}}\,dx$

Can you use hyperbolic substitution?

$$\text{Let } x = 2\tan\theta \Rightarrow dx = 2\sec^2\theta\, d\theta$$

$$\therefore \sqrt{4+x^2} = \sqrt{4+4\tan^2\theta} = 2\sqrt{\sec^2\theta} = 2\sec\theta$$

$$\because -\pi/2 \le \theta \le \pi/2 \Rightarrow \sec\theta \ge 0$$

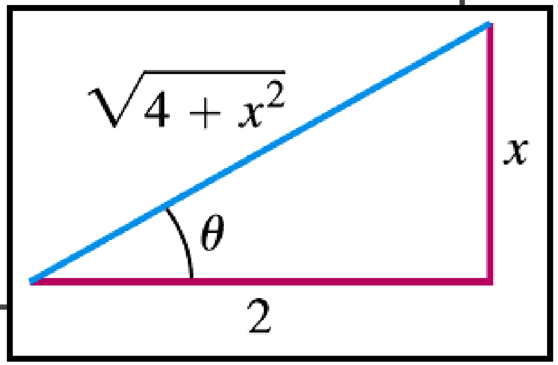
$$\therefore I = \int \frac{1}{x^2\sqrt{4+x^2}}\,dx = \int \frac{2\sec^2\theta}{4\tan^2\theta . 2\sec\theta}\,d\theta = \frac{1}{4}\int \frac{\sec\theta}{\tan^2\theta}\,d\theta$$

$$I = \frac{1}{4}\int \frac{1}{\cos\theta}.\frac{\cos^2\theta}{\sin^2\theta}\,d\theta = \frac{1}{4}\int \frac{\cos\theta}{\sin^2\theta}\,d\theta$$

$$\text{Let } u = \sin\theta \Rightarrow du = \cos\theta\, d\theta$$

$$\therefore I = \frac{1}{4}\int \frac{du}{u^2} = \frac{-1}{4}\frac{1}{u} + C = \frac{-1}{4}\frac{1}{\sin\theta} + C$$

$$\therefore I = \frac{-1}{4}\frac{\sqrt{4+x^2}}{x} + C$$



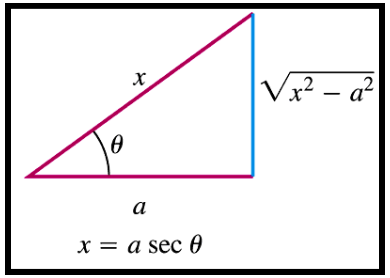
Example 4: *Find the value of* $I=\int \frac{x^2}{\sqrt{x^2-a^2}}dx,\ a>0$

Let $x = a\sec\theta \Rightarrow dx = a\sec\theta\tan\theta d\theta$

$$\therefore \sqrt{x^2-a^2} = \sqrt{a^2\sec^2\theta - a^2} = a\sqrt{\tan^2\theta} = a\tan\theta$$

$$\therefore I = \int \frac{x^2}{\sqrt{x^2-a^2}}dx = \int \frac{a^3\sec^3\theta\tan\theta}{a\tan\theta}d\theta = a^2\int \sec^3\theta d\theta$$

$$I = \frac{a^2}{2}[\sec\theta\tan\theta + \ln|\sec\theta+\tan\theta|] + C$$



$$= \frac{a^2}{2}\left[\frac{\sqrt{x^2-a^2}}{a}.\frac{x}{a} + \ln\left|\frac{\sqrt{x^2-a^2}}{a}+\frac{x}{a}\right|\right] + C$$

Example 5: *Find the value of* $I=\int \frac{x}{\sqrt{3-2x-x^2}}\,dx$

$$3-2x-x^2=-(x^2+2x-3)=-(x+1)^2+4=4-(x+1)^2$$

$$\text{Let } u=x+1 \Rightarrow du=dx,\ x=u-1$$

$$\therefore I=\int \frac{x}{\sqrt{3-2x-x^2}}\,dx=\int \frac{u-1}{\sqrt{4-u^2}}\,dx$$

You can also finish it directly without further substitutions. How?

$$\text{Now let, } u=2\sin\theta \Rightarrow du=2\cos\theta\, d\theta$$

$$\therefore I=\int \frac{2\sin\theta-1}{2\cos\theta}.2\cos\theta\, d\theta=\int(2\sin\theta-1)\,d\theta=-2\cos\theta-\theta+C$$

$$\therefore I=-\sqrt{4-u^2}-\sin^{-1}\left(\frac{u}{2}\right)+C=-\sqrt{4-(x+1)^2}-\sin^{-1}\left(\frac{x+1}{2}\right)+C$$

Example 6: Find the value of $I=\int\sqrt{\frac{x-1}{x+1}}dx$

$$I=\int\sqrt{\frac{x-1}{x+1}}dx=\int\frac{\sqrt{x-1}}{\sqrt{x+1}}dx=\int\frac{\sqrt{x-1}}{\sqrt{x+1}}.\frac{\sqrt{x-1}}{\sqrt{x-1}}dx=\int\frac{x-1}{\sqrt{x+1}\sqrt{x-1}}dx$$

$$=\int\frac{x-1}{\sqrt{(x+1)(x-1)}}dx=\int\frac{x-1}{\sqrt{x^2-1}}dx=\int\frac{x}{\sqrt{x^2-1}}dx-\int\frac{1}{\sqrt{x^2-1}}dx$$

$$=\int x(x^2-1)^{-1/2}dx-\cosh^{-1}x=\frac{1}{2}\int(x^2-1)^{-1/2}(2x)dx-\cosh^{-1}x$$

$$=\sqrt{x^2-1}-\cosh^{-1}x+C$$

Example 6: *Find the value of* $I=\int\sqrt{\frac{x-2}{x+4}}dx$

$$I=\int\sqrt{\frac{x-2}{x+4}}dx=\int\frac{\sqrt{x-2}}{\sqrt{x+4}}dx=\int\frac{\sqrt{x-2}}{\sqrt{x+4}}.\frac{\sqrt{x-2}}{\sqrt{x-2}}dx=\int\frac{x-2}{\sqrt{(x+4)(x-2)}}dx$$

$$=\int\frac{x-2}{\sqrt{x^2+2x-8}}dx=\int\frac{x-2}{\sqrt{(x+1)^2-9}}dx=\int\frac{u-3}{\sqrt{u^2-9}}dx,\quad u=x+1$$

$$=\int\frac{u}{\sqrt{u^2-9}}dx-\int\frac{3}{\sqrt{u^2-9}}dx$$

$$=\sqrt{u^2-9}\;-3\int\frac{1}{\sqrt{u^2-9}}dx$$

$$=\sqrt{x^2+2x-8}-3\cosh^{-1}\left(\frac{u}{3}\right)=\sqrt{x^2+2x-8}-3\cosh^{-1}\left(\frac{x+1}{3}\right)+C$$