Math 2

Lecture 4

Basic integration techniques

Integration by parts

Objectives

1. *Introduce the technique of integration by parts.*
2. *Learn when to use this technique.*
3. *Use integration by parts to obtain a reduction formula for certain integrals.*


IF YOU HAVE
A PROBLEM,
face IT...
DON'T
facebook
IT!!!!!!!!!!!!!!!!!!

Integration by Parts

- This technique is particularly useful for integrals involving the products of algebraic and exponential , logarithmic, or some trigonometric functions (sin x or cos x), such as

$$\int x e^x \, dx, \qquad \int x^2 \ln x \, dx, \qquad \int x \sin x \, dx$$

- Integration by parts is based on the product rule for differentiation

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$uv = \int u\frac{dv}{dx}dx + \int v\frac{du}{dx}dx \quad (Integrate\ each\ side)$$

$$uv = \int u\,dv + \int v\,du \quad (Write\ in\ differential\ form)$$

$$\Rightarrow \int u\,dv = uv - \int v\,du$$

- Integration by Parts :
- Let u and v be differentiable functions of x.

$$\int u\,dv = uv - \int v\,du$$

- Note that the formula for integration by parts expresses the original integral in terms of another integral. Depending on the choices of u and dv, it may be easier to evaluate the second integral than the original one.

- Example 1: Find $\int xe^{x}dx$

- Solution

Let $u=x,$ then $du=1$

and $dv = e^{x}dx$ then $v = e^{x}$

$$\int xe^{x}dx = (e^{x})x - \int e^{x}.1dx = xe^{x} - e^{x} + C$$

- Example 2: Find $\int x\sin(2x)\,dx$

Solution: We assume that u=x and dv= sin(2x) dx, then $u = x \qquad dv = \sin(2x)\,dx$

$$\Rightarrow du = dx \qquad v = \int \sin(2x)\,dx = -\frac{1}{2}cox(2x)$$

$$Formula: \int u\,dv = uv - \int v\,du$$

$$\Rightarrow \int x\sin(2x)\,dx = x(-\frac{1}{2}\cos(2x)) - \int -\frac{1}{2}\cos(2x)\,dx$$

$$\Rightarrow \int x\sin(2x)\,dx = -\frac{x}{2}\cos(2x) + \frac{1}{4}\sin(2x) + C$$

Note that If you assumed that u=sin(2x) and v=x dx, the integration will not be found easily. In fact the second integral will be more complicated than the first one Please try doing that.
That is raise the question? What are the rules of choosing u and dv?

Guidelines for choosing u and dv

1. Let dv be the most complicated portion of the integrand that fits a basic integration formula. Let u be the remaining factor
2. Let u be the portion of the integrand whose derivative is a function simpler than u. Let dv be the remaining factor

Example 3: Find $\int x^2 e^{-x}\,dx$

Solution

$$Let \quad u = x^2 \qquad dv = e^{-x}\,dx$$

$$Then \quad du = 2x\,dx \qquad v = -e^{-x}$$

$$Formula: \quad \int u\,dv = uv - \int v\,du$$

$$\Rightarrow \quad \int x^2 e^{-x}\,dx = -x^2 e^{-x} + 2\int x e^{-x}\,dx \qquad (1)$$

$$We \;\; Integrate \quad \int x e^{-x}\,dx \quad by \;\; parts \;\; again$$

$$Let \quad u = x \qquad dv = e^{-x}\,dx$$

$$\Rightarrow \quad du = dx \qquad v = -e^{-x}$$

$$\int x e^{-x}\,dx = -x e^{-x} + \int e^{-x}\,dx = -x e^{-x} - e^{-x} \qquad (2)$$

$$Subst.\ (2) \;\; in \;\; (1)$$

$$\Rightarrow \quad \int x^2 e^{-x}\,dx = -x^2 e^{-x} + 2(-x e^{-x} - e^{-x})$$

$$= -e^{-x}(x^2 + 2x + 2) + C$$

Integration by parts: Case 2 (Integrand with a single term)

Although the integrand in this case is a single term and not a product of two functions, integration by parts is used

Example 4:

Find $\int \ln x \, dx$

Solution:

$$Let \quad u = \ln x \qquad\qquad dv = dx$$

$$\Rightarrow du = \frac{1}{x} dx \qquad\qquad v = x$$

$$\Rightarrow \int \ln x dx = x \ln x - \int x \frac{1}{x} dx = x \ln x - x + C$$

Example 5: Find the integration

(1) $\int \tan^{-1} x dx$

(2) $\int x^2 \tan^{-1} x dx$

(3) $\int \sin^{-1} x dx$

(4) $\int \sec^{-1} x dx$

Example 5: Find the integration

(1) $\int \tan^{-1} x dx$ (2) $\int x^2 \tan^{-1} x dx$

(3) $\int \sin^{-1} x dx$ (4) $\int \sec^{-1} x dx$

Solution:

(1) $\int \tan^{-1} x dx$

$$= \int 1.\tan^{-1} x dx$$

$$= x\tan^{-1} x - \int x.\frac{1}{1+x^2} dx$$

$$= x\tan^{-1} x - \frac{1}{2}\ln(1+x^2) + C$$

Example 5: Find the integration

(1) $\int \tan^{-1} x dx$ (2) $\int x^2 \tan^{-1} x dx$

(3) $\int \sin^{-1} x dx$ (4) $\int \sec^{-1} x dx$

Solution:

(2) $\int x^2 \tan^{-1} x dx$

$$= \frac{x^3}{3}\tan^{-1} x - \frac{1}{3}\int x^3 . \frac{1}{1+x^2} dx$$

$$= \frac{x^3}{3}\tan^{-1} x - \frac{1}{3}\int (x - \frac{x}{1+x^2}) dx$$

$$= \frac{x^3}{3}\tan^{-1} x - \frac{1}{6}x^2 + \frac{1}{6}\ln(1+x^2) + C$$

Example 5: Find the integration

(1) $\int \tan^{-1} x dx$ (2) $\int x^2 \tan^{-1} x dx$

(3) $\int \sin^{-1} x dx$ (4) $\int \sec^{-1} x dx$

Solution:

(3) $\int \sin^{-1} x dx$

$$= x\sin^{-1} x - \int x . \frac{1}{\sqrt{1-x^2}} dx$$

$$= x\sin^{-1} x + \frac{1}{2}\int (1-x^2)^{-1/2}(-2x)dx$$

$$= x\sin^{-1} x + \sqrt{1-x^2} + C$$

Example 5: Find the integration

(1) $\int \tan^{-1} x dx$ (2) $\int x^2 \tan^{-1} x dx$

(3) $\int \sin^{-1} x dx$ (4) $\int \sec^{-1} x dx$

Solution:

(4) $\int \sec^{-1} x dx$

$$= x\sec^{-1} x - \int x.\frac{1}{x\sqrt{x^2-1}} dx$$

$$= x\sec^{-1} x - \int \frac{1}{\sqrt{x^2-1}} dx$$

$$= x\sec^{-1} x - \cosh^{-1} x + C$$

Repeated integration by parts with a Twist: Case 3

In this case when we apply integration by parts once or twice we return back to the original integral ,hence we can use this idea to evaluate certain Integrals as in the following example:

Repeated integration by parts with a Twist: Case 3

Example 6: Evaluate $\int e^{2x} \sin x \, dx$

Solution

$$\text{Let } u = e^{2x} \text{ and } dv = \sin x$$

$$\text{Then } du = 2e^{2x}\,dx \quad \text{and} \quad v = -\cos x$$

$$I = \int e^{2x} \sin x \, dx = -e^{2x} \cos x + 2\int e^{2x} \cos x \, dx$$

$$= -e^{2x} \cos x + 2[e^{2x} \sin x - 2\int e^{2x} \sin x \, dx]$$

$$= -e^{2x} \cos x + 2e^{2x} \sin x - 4\int e^{2x} \sin x \, dx$$

$$= -e^{2x} \cos x + 2e^{2x} \sin x - 4I$$

$$\therefore I + 4I = -e^{2x} \cos x + 2e^{2x} \sin x$$

$$\therefore 5I = -e^{2x} \cos x + 2e^{2x} \sin x$$

$$\therefore I = \frac{1}{5}[-e^{2x} \cos x + 2e^{2x} \sin x] + C$$

Integration using reduction formula

- *We mean by a reduction formula for an integral is to express an integral involving powers of certain expression in terms of another integral involving lower powers of that expression. The following example explains how to do that*

Integration using reduction formula

Example 7: Find a reduction formula for the integral $I_n = \int \cos^n x \, dx$, then use it to find $\int \cos^4 x \, dx$.

Solution :

$$I_n = \int \cos^n x \, dx = \int \cos x . \cos^{n-1} x \, dx$$

$$= \sin x \cos^{n-1} x - (n-1) \int \sin x . \cos^{n-2} x (-\sin x) dx$$

$$= \sin x \cos^{n-1} x + (n-1) \int \sin^2 x . \cos^{n-2} x \, dx$$

$$= \sin x \cos^{n-1} x + (n-1) \int (1 - \cos^2 x) \cos^{n-2} x \, dx$$

$$= \sin x \cos^{n-1} x + (n-1) \int \cos^{n-2} x \, dx - (n-1) \int \cos^n x \, dx$$

$$= \sin x \cos^{n-1} x + (n-1) I_{n-2} - (n-1) I_n$$

$$\therefore I_n + (n-1) I_n = \sin x \cos^{n-1} x + (n-1) I_{n-2}$$

$$\therefore nI_n = \sin x \cos^{n-1} x + (n-1) I_{n-2}$$

$$\therefore I_n = \frac{1}{n} (\sin x \cos^{n-1} x + (n-1) I_{n-2})$$

From the definition of I_n : $I_0 = \int \cos^0 x dx = \int 1 dx = x$

$\int \cos^4 x dx = I_4$, then using the reduction formula we get

$$n = 2 \Rightarrow I_2 = \frac{1}{2}(\sin x \cos x + I_0) = \frac{1}{2}(\sin x \cos x + x)$$

$$n = 4 \Rightarrow I_4 = \frac{1}{4}(\sin x \cos^3 x + 3I_2)$$

$$\therefore I_4 = \frac{1}{4}[\sin x \cos^3 x + \frac{3}{2}(\sin x \cos x + x)] + C$$