Computer Control of Dynamic Systems

Lecture-2

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CONTROLLABILITY

- A control system is controllable if every state variable can be controlled in a finite time period by some unconstrained control signal.
- If any state variable is independent of the control signal, then it is impossible to control this state variable and therefore the system is uncontrollable.
- For

$$x\,(k+1) = A\,x\,(k) + B\,u(k)$$
$$y(k) = C\,x\,(k)$$

$$Q_c = [B\ AB\ \ldots\ldots\ldots A^{n-1}B]$$

- If

$$|Q_c| \begin{cases} \neq 0 & Controllable \\ = 0 & Uncontrollable \end{cases}$$

OBSERVABILITY

- The system, therefore, is completely observable if every transition of the state eventually affects every element of the output vector.
- For

$$x\,(k+1) = A\,x\,(k) + B\,u(k)$$
$$y(k) = C\,x\,(k)$$

$$Q_o = \begin{bmatrix} C \\ CA \\ . \\ . \\ . \\ CA^{n-1} \end{bmatrix}$$

- If

$$|Q_o| \begin{cases} \neq 0 & Observable \\ = 0 & Unobservable \end{cases}$$

Solution of discrete-time state space equations

The Recursive solution:

- We seek an expression for the state at time k in terms of the initial condition vector x(0) and the input sequence u(k).
- Given

$$x\,(k+1) = A\,x\,(k) + B\,u(k)$$
$$y(k) = C\,x\,(k)$$

then:

$$x\,(1) = A\,x\,(0) + B\,u(0)$$
$$x\,(2) = A\,x\,(1) + B\,u(1)$$
$$= A^2\,x\,(0) + AB\,u(0) + B\,u(1)$$
$$x\,(3) = A^3\,x\,(0) + A^2 B\,u(0) + AB\,u(1) + B\,u(2)$$

.

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$$x\,(n) = A^n\,x\,(0) + A^{n-1}B\,u(0) + A^{n-2}B\,u(1) + \cdots + AB\,u(n-2) + B\,u(n-1)$$

$$x(1) = A\,x(0) + B\,u(0)$$

$$x(2) = A\,x(1) + B\,u(1)$$

$$= A^2\,x(0) + AB\,u(0) + B\,u(1)$$

$$x(3) = A^3\,x(0) + A^2B\,u(0) + AB\,u(1) + B\,u(2)$$

.

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$$x(n) = A^n\,x(0) + A^{n-1}B\,u(0) + A^{n-2}B\,u(1) + \cdots + AB\,u(n-2) + B\,u(n-1)$$

$$x(n) = A^n\,x(0) + \sum_{i=0}^{n-1} A^{n-i-1}B\,u(i)$$

Solution using Z-domain

- Given

$$X(k+1) = A\,X(k) + B\,u(k)$$

Then

$$ZX(z) - ZX(0) = A\,X(z) + B\,u(z)$$
$$(ZI - 1)X(z) = ZX(0) + B\,u(z)$$

$$X(z) = Z\,(ZI - A)^{-1}X(0) + (ZI - A)^{-1}B\,u(z)$$

$$X(z) = \varphi(z)X(0) + (ZI - A)^{-1}B\,u(z)$$

Where $\varphi(z)$=$Z\,(ZI - A)^{-1}$

$$\mathcal{Z}^{-1}[\varphi(z)]=\varphi(k)$$

$$\boxed{X(k) = \varphi(k)X(0) + \varphi(k-1)B\,u(z)}$$

Example

- Given the state space representation of a control system as:

$$X(k+1) = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} X(k) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(k)$$

find the solution of a state equations for a unit step input and $X(0) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

- Solution:

$$X(k+1) = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} X(k) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(k)$$

$X(0) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, *u(k) =1 for k=0,1,2,.....*

$$X(k) = \varphi(k) X(0) + \sum_{i=0}^{n-1} \varphi(k-1-i)\, B\, u(i)$$

$$\varphi(k) = \mathcal{Z}^{-1}[Z\,(ZI - A)^{-1}]$$

$$(ZI - A)^{-1} = \begin{bmatrix} Z-1 & -1 \\ 0 & Z+1 \end{bmatrix}^{-1} = \frac{1}{Z^2-1} \begin{bmatrix} Z-1 & -1 \\ 0 & Z+1 \end{bmatrix}$$

$$Z(ZI-A)^{-1} = \frac{z}{(Z+1)(Z-1)}\begin{bmatrix} Z-1 & -1 \\ 0 & Z+1 \end{bmatrix} = \begin{bmatrix} \frac{Z}{(Z-1)} & \frac{Z}{(Z+1)(Z-1)} \\ 0 & \frac{Z}{(Z+1)} \end{bmatrix}$$

$$\varphi(k) = \mathcal{Z}^{-1}[Z\,(ZI-A)^{-1}] = \mathcal{Z}^{-1}\begin{bmatrix} \frac{Z}{(Z-1)} & Z(\frac{\frac{1}{2}}{(Z-1)} - \frac{\frac{1}{2}}{(Z+1)} \\ 0 & \frac{Z}{(Z+1)} \end{bmatrix}$$

$$\varphi(k) = \begin{bmatrix} 1^k & \frac{1}{2}(1^k-(-1)^k) \\ 0 & (-1)^k \end{bmatrix}$$

$$X(k)=\begin{bmatrix} 1^k & \frac{1}{2}(1^k-(-1)^k) \\ 0 & (-1)^k \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix}+$$

$$\sum_{i=0}^{k-1}\begin{bmatrix} 1^{k-1-i} & \frac{1}{2}(1^{k-1-i}-(-1)^{k-1-i}) \\ 0 & (-1)^{k-1-i} \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\therefore X(1) = \Phi(1)\, X(0) + \Phi(0)\, B$$

$$= \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

$$\& \; X(2) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ +1 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

& so on, we can get $X(3)$, $X(4)$, ...

OR Another solution :- By using recursion.

For $K = 0$

$$\therefore X(1) = A\,X(0) + \underline{b}\,u(0) = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

& For $K = 1$

$$\therefore X(2) = A\,X(1) + b\,u(1) = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 0 \\ 2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

& so on, we can get $x(3)$, $x(4)$, —

End Of Lecture