Computer Control of Dynamic Systems

Lecture-9,10

Least Squares System Identification

Emam Fathy
email: emfmz@aast.edu

Introduction

- The frequency-response techniques and the root-locus technique are classical design techniques (based on the transfer function).

- They are very effective, but are largely trial and error.

- Even when an acceptable design is completed, the question remains as to whether a "better" design could be found.

Introduction

- The pole-assignment design technique is termed a modern technique (based on the state variable model of the plant).

- In this procedure we assumed that we know the exact locations required for the closed-loop transfer-function poles, and we can realize these locations, at least in the linear model.

- **For the physical system**, the regions in which the pole locations can be placed are limited.

- In the pole-assignment technique, we **assume** that we know the pole locations that yield the "best" control system.

Introduction

- We need a different technique that yields the "best" control system.

- This technique is an optimal design technique, and assumes that we can write a mathematical function which is called the *cost function.*

- The optimal design procedure minimizes this cost function: hence the term *optimal.*

- The final two topics presented in this course are based on the same mathematical foundation as for linear quadratic optimal control.

Introduction

- **The first topic** is a technique for system identification.

- The system transfer function is calculated from input-output data for the physical system.

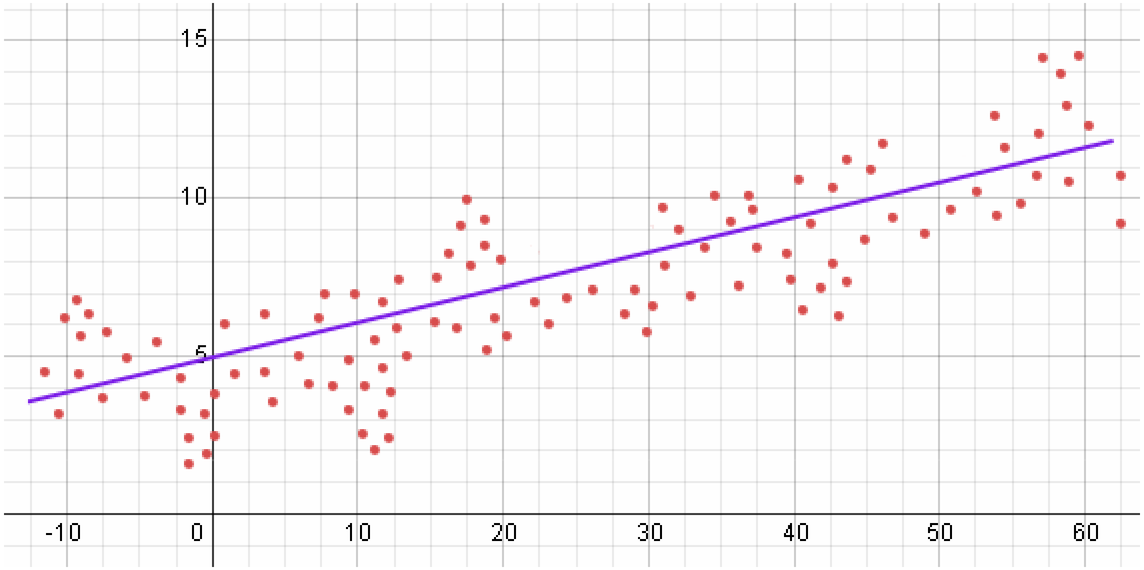
- This technique is called least-squares system identification, and yields the transfer function that "best" fits the available data.

- **The second topic** is Kalman filtering, is an optimal technique for state estimation.

Least Squares Method

Least Squares Curve Fitting

- Many different techniques are available for finding a linear model of a physical system by using input-output measurements.



- We will consider least-squares system identification.

- Suppose that we suspect a linear relationship between the variables x and y of the form

$$y = kx \quad .....(1)$$

k: constant

- We measure data pairs (X_i, Y_i) and wish to calculate the "best" estimate of k from the data.

- Using the data pairs, (1) can be expressed as,

$$\begin{aligned} Y_1 &= k x_1 + e_1 \\ Y_2 &= k x_2 + e_2 \\ Y_3 &= k x_3 + e_3 \\ &\vdots \end{aligned}$$

e: errors in measuring data

- If no errors are present, we can determine k exactly from anyone data pair.

- We wish to solve for k by a method that minimizes the errors.

- We can write (1) in vector form as

$$y = k x + e$$

- For three data pairs

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad \mathbf{e} = \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix}$$

- For N data pairs

$$\mathbf{e}^T\mathbf{e} = e_1^2 + e_2^2 + \cdots + e_N^2 = \sum_{k=1}^{N} e_k^2$$

- The sum of the squared errors is given by

$$\mathbf{e}^T\mathbf{e} = [\mathbf{y} - k\mathbf{x}]^T[\mathbf{y} - k\mathbf{x}] = \mathbf{y}^T\mathbf{y} - 2k\mathbf{x}^T\mathbf{y} + k^2\mathbf{x}^T\mathbf{x}$$

- The resulting estimate of *k* is called the *least-squares estimate.*

- We obtain the least-squares estimate of *k* from

$$\frac{\partial(\mathbf{e}^T \mathbf{e})}{\partial k} = -2\mathbf{x}^T \mathbf{y} + 2k\mathbf{x}^T \mathbf{x} = 0$$

- Solving this equation for k yields

$$\hat{k} = (\mathbf{x}^T \mathbf{x})^{-1} \mathbf{x}^T \mathbf{y} = \frac{\mathbf{x}^T \mathbf{y}}{\mathbf{x}^T \mathbf{x}}$$

- where $\hat{k}$ is the least-squares estimate of *k*

Example

Suppose that $y = k\,x$ and we wish to determine the least squares estimate of k from the three data pairs

x	y
1.0	1.25
2.1	2.0
2.95	2.9

$$\hat{k} = \frac{(1.0)(1.25) + (2.1)(2.0) + (2.95)(2.9)}{(1.0)(1.0) + (2.1)(2.1) + (2.95)(2.95)} = \frac{14.005}{14.1125} = 0.9924$$

LEAST-SQUARES SYSTEM IDENTIFICATION

- We assume a system transfer-function model of the form

$$\frac{Y(z)}{U(z)} = G(z) = \frac{b_1 z^{n-1} + b_2 z^{n-2} + \cdots + b_n}{z^n - a_1 z^{n-1} - \cdots - a_n}$$

- The difference equation of the system is:

$$y(k) = a_1 y(k-1) + a_2 y(k-2) + \cdots + a_n y(k-n)$$
$$+ b_1 u(k-1) + b_2 u(k-2) + \cdots + b_n u(k-n)$$

- We wish to determine the coefficient vector from measurements of the input-output sequences *u(k)* and *y(k).*

$$\theta = (a_1 \quad a_2 \cdots a_n \quad b_1 \quad b_2 \cdots b_n)^T$$

- To illustrate the procedure, we first consider the first-order case, with

$$\frac{Y(z)}{U(z)} = G(z) = \frac{b_1}{z - a_1}$$

Hence

$$y(k) = a_1 y(k-1) + b_1 u(k-1)$$

and thus

$$y(1) = a_1 y(0) + b_1 u(0) + e(1)$$

$$y(2) = a_1 y(1) + b_1 u(1) + e(2)$$

$$y(3) = a_1 y(2) + b_1 u(2) + e(3)$$

- e(k): error due to measurement inaccuracies.

- This equation can be expressed in vector-matrix form as

$$\begin{bmatrix} y(1) \\ y(2) \\ y(3) \end{bmatrix} = \begin{bmatrix} y(0) & u(0) \\ y(1) & u(1) \\ y(2) & u(2) \end{bmatrix} \begin{bmatrix} a_1 \\ b_1 \end{bmatrix} + \begin{bmatrix} e(1) \\ e(2) \\ e(3) \end{bmatrix}$$

- which may be expressed as the

$$y(3) = F(3)\,\theta + e(3)$$

- ***For a set of (N + 1) measurement pairs***

$$\{u(0), y(0)\}, \{u(1), y(1)\}, \ldots, \{u(N), y(N)\}$$

- Define the vector f(k) by

$$\mathbf{f}^T(k) = [y(k-1) \quad y(k-2) \cdots y(k-n) \quad u(k-1) \cdots u(k-n)]$$

then

$$\begin{aligned} y(n) &= \mathbf{f}^T(n)\boldsymbol{\theta} + e(n) \\ y(n+1) &= \mathbf{f}^T(n+1)\boldsymbol{\theta} + e(n+1) \\ &\vdots \\ y(N) &= \mathbf{f}^T(N)\boldsymbol{\theta} + e(N) \end{aligned}$$

$$y(N) = \mathbf{f}^T(N)\boldsymbol{\theta} + e(N)$$

- where

$$\mathbf{y}(N) = \begin{bmatrix} y(n) \\ y(n+1) \\ \vdots \\ y(N) \end{bmatrix}, \quad \mathbf{F}(N) = \begin{bmatrix} \mathbf{f}^T(n) \\ \mathbf{f}^T(n+1) \\ \vdots \\ \mathbf{f}^T(N) \end{bmatrix}, \quad \mathbf{e}(N) = \begin{bmatrix} e(n) \\ e(n+1) \\ \vdots \\ e(N) \end{bmatrix}$$

- we can express *y(N)* as:

$$\mathbf{y}(N) = \mathbf{F}(N)\boldsymbol{\theta} + \mathbf{e}(N)$$

- Next the cost function $J(\theta)$ is defined as the sum of the squared errors:

$$J(\boldsymbol{\theta}) = \sum_{k=n}^{N} e^2(k) = \mathbf{e}^T(N)\mathbf{e}(N)$$

- Then

$$J(\boldsymbol{\theta}) = [\mathbf{y} - \mathbf{F}\boldsymbol{\theta}]^T[\mathbf{y} - \mathbf{F}\boldsymbol{\theta}] = \mathbf{y}^T\mathbf{y} - \boldsymbol{\theta}^T\mathbf{F}^T\mathbf{y} - \mathbf{y}^T\mathbf{F}\boldsymbol{\theta} + \boldsymbol{\theta}^T\mathbf{F}^T\mathbf{F}\boldsymbol{\theta}$$

$$= \mathbf{y}^T\mathbf{y} - 2\boldsymbol{\theta}^T\mathbf{F}^T\mathbf{y} + \boldsymbol{\theta}^T\mathbf{F}^T\mathbf{F}\boldsymbol{\theta}$$

- Thus the value of θ that minimizes $J(\theta)$ satisfies the equation

$$\frac{\partial J(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}} = -2\mathbf{F}^T\mathbf{y} + 2\mathbf{F}^T\mathbf{F}\boldsymbol{\theta} = \mathbf{0} \qquad \mathbf{F}^T\mathbf{F}\boldsymbol{\theta} = \mathbf{F}^T\mathbf{y}$$

$$\mathbf{F}^T\mathbf{F}\boldsymbol{\theta} = \mathbf{F}^T\mathbf{y}$$

- Then The least-squares estimate of θ is then

$$\hat{\boldsymbol{\theta}}_{LS} = [\mathbf{F}^T(N)\mathbf{F}(N)]^{-1}\mathbf{F}^T(N)\mathbf{y}(N)$$

- For N=4

$$\mathbf{F}^T(4)\mathbf{y}(4) = \begin{bmatrix} y(1) & y(0) & u(1) & u(0) \\ y(2) & y(1) & u(2) & u(1) \\ y(3) & y(2) & u(3) & u(2) \end{bmatrix}^T \begin{bmatrix} y(2) \\ y(3) \\ y(4) \end{bmatrix}$$

Example

Suppose that a first-order system yields the following data.

k	$u(k)$	$y(k)$
0	1.0	0
1	0.75	0.3
2	0.50	0.225

- Assumed transfer function is

$$G(z) = \frac{b_1}{z - a_1}, \qquad \boldsymbol{\theta} = \begin{bmatrix} a_1 \\ b_1 \end{bmatrix}$$

- Since N=2

$$\mathbf{F}(2) = \begin{bmatrix} \mathbf{f}^T(1) \\ \mathbf{f}^T(2) \end{bmatrix} = \begin{bmatrix} y(0) & u(0) \\ y(1) & u(1) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0.3 & 0.75 \end{bmatrix}$$

- Then

$$\mathbf{F}^T\mathbf{F} = \begin{bmatrix} 0 & 0.3 \\ 1 & 0.75 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0.3 & 0.75 \end{bmatrix} = \begin{bmatrix} 0.09 & 0.225 \\ 0.225 & 1.5625 \end{bmatrix}$$

- and $$[\mathbf{F}^T\mathbf{F}]^{-1} = \begin{bmatrix} 17.361 & -2.5 \\ -2.5 & 1 \end{bmatrix}$$

- Then the least-squares estimate of θ is

$$\hat{\boldsymbol{\theta}}_{LS} = [\mathbf{F}^T\mathbf{F}]^{-1}\mathbf{F}^T\mathbf{y} = \begin{bmatrix} 17.361 & -2.5 \\ -2.5 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0.3 \\ 1 & 0.75 \end{bmatrix} \begin{bmatrix} 0.3 \\ 0.225 \end{bmatrix}$$

$$= \begin{bmatrix} -2.5 & 3.333 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0.3 \\ 0.225 \end{bmatrix} = \begin{bmatrix} 0 \\ 0.3 \end{bmatrix} = \begin{bmatrix} a_1 \\ b_1 \end{bmatrix}$$

- Then the transfer function is:

$$G(z) = \frac{0.3}{z}$$

- with the difference equation

$$y(k) = 0.3u(k - 1)$$

End Of Lecture