# Boundary Value and Initial Condition Problems in Partial Differential Equations:

# Wave Equation

Lecture 4

*Dr. Amr Bayoumi*

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# Outline

- Wave Equation
  - Boundary Values
  - Initial Condition
- Separation of Variables
- Fourier Series Analysis

# Wave Equation (PDE)

$$\frac{\partial u^2(x,t)}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$$

Example: Electromagnetic Waves:

$$(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2})\overrightarrow{E(x,t)}=0 \qquad (\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2})\overrightarrow{B(x,t)}=0$$

**Boundary Conditions** (u(x,t) at certain x's, at any t):

$$u(0,t) = 0, \qquad u(L,t) = 0$$

**Initial Conditions** (u(x,t) or its derivatives at t=0, at any x):

$$u(x,0) = f(x) \qquad \frac{\partial u(x,0)}{\partial t} = g(x)$$

Examples: Vibrating string with fixed ends, EM standing waves

# Separation of Variables

Let: $\boldsymbol{u(x,t) = F(x)\, G(t)}$

$$\frac{\partial^2 u}{\partial t^2} = F\ddot{G} \quad , \frac{\partial^2 u}{\partial x^2} = G\ F''$$

Substitute in Wave PDE:

$$F\ddot{G} = c^2\ G\ F'' \quad \rightarrow \quad \frac{F''}{F} = \frac{\ddot{G}}{c^2 G} = k = constant$$

$k$= constant since $F''/F$ does not depend on $t$ , and $\ddot{G}/G$ does not depend on $x$, thus $\frac{\partial}{\partial x}$ $and \frac{\partial}{\partial t}$ $of\ RHS\ \&\ LHS\ is\ zero$



$$\boldsymbol{F'' - k\,F = 0 \quad , \ddot{G} - c^2 k\, G = 0}$$

# Boundary Conditions:

$$F'' - k\,F = 0$$

For the Differential Eqn: $F'' = kF$

$$F(x) = C_1\, e^{\sqrt{k}x} + C_2 e^{-\sqrt{k}x}$$

Case $k=0$:

$$F(x) = C_1 + C_2$$

Using BC:

$F(0) = 0 = C_1 + C_2 = F(x) \rightarrow F(x)G(t) = 0$ *(Unacceptable)*

Case k= +ve = $\mu^2$:

$F(0) = 0 = C_1 + C_2 \rightarrow F(x) = C_1(e^{\mu x} - e^{-\mu x})$ =sinh$(\mu x)$

$$F(L) = 0 \rightarrow \sinh(\mu L) \neq 0$$



Since $\sinh(\theta) = 0$ *ONLY at* θ=0 -> k *cannot* be positive

Therefore: $k = -ve = -p^2$

# Boundary Conditions:

$$F'' - k F = 0$$

$k = -ve = -p^2$: $F(x) = C_1 e^{jpx} + C_2 e^{-jpx}$

Using BC: $F(0) = 0 = C_1 + C_2$

Using: $e^{j\theta} = \cos\theta + j\sin\theta$ -> $F(x) = 2jC_1 \sin(px)$

Normalizing $1 = 2jC_1$ -> $F(x) = \sin(px)$

Using BC: $F(L) = \sin(pL) = 0 \rightarrow pL = \pm n\pi \; (n = 1,2,3..)$

$F_n(x) = \sin(\frac{n\pi}{L}x)$

Using Superposition of independent solutions of ODEs:

$$F(x) = \sum_{n=1}^{\infty} F_n(x) = \sum_{n=1}^{\infty} \sin(\frac{n\pi}{L}x)$$

# Initial Conditions & Fourier Series in PDE:

$$\ddot{G} - c^2 k\, G = 0$$

ODE for any n: $\ddot{G}_n + c^2 {p_n}^2 G_n = 0 \rightarrow \ddot{G}_n + {\lambda_n}^2 G_n = 0$

$$\lambda_n = c\, p_n = c\frac{n\pi}{L}$$

General Solution for $G_n(t)$ : $G_n(t) = A_n \cos(\lambda_n t) + Bn \sin(\lambda_n t)$

Overall Soln.:

$$u(x,t) = \sum_{n=1}^{\infty} F_n(x) G_n(t) = \sum_{n=1}^{\infty} [A_n \cos(\lambda_n t) + Bn \sin(\lambda_n t)] \sin(\frac{n\pi}{L} x)$$

**Initial Conditions (*t=0*):**

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} [A_n \cos(\lambda_n 0) + Bn \sin(\lambda_n 0)] \sin(\frac{n\pi}{L} x)$$

$$\sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{L} x\right) = f(x)$$

Using Fourier Series: $A_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi}{L} x\right) dx$

# Initial Conditions & Fourier Series in PDE (2):

$$\ddot{G} - c^2 k\, G = 0$$

Overall Soln.:

$$u(x,t) = \sum_{n=1}^{\infty} F_n(x) G_n(t) = \sum_{n=1}^{\infty} [A_n \cos(\lambda_n t) + Bn \sin(\lambda_n t)] \sin(\frac{n\pi}{L} x)$$

Initial Conditions ($t=0$): $\frac{\partial u(x,0)}{\partial t} = g(x)$

$$\sum_{n=1}^{\infty} [-A_n \lambda_n \sin(\lambda_n t) + Bn\, \lambda_n \cos(\lambda_n t)] \sin\left(\frac{n\pi}{L} x\right) = g(x)$$

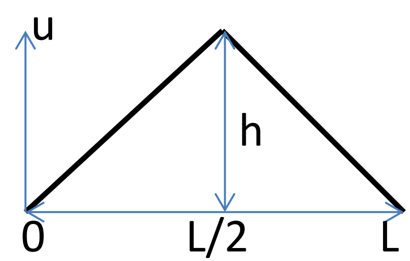
$$\sum_{n=1}^{\infty} B_n \lambda_n \sin\left(\frac{n\pi}{L} x\right) = g(x)$$

Using Fourier Series: $B_n \lambda_n = \frac{2}{L} \int_0^L g(x) \sin\left(\frac{n\pi}{L} x\right) dx$

# Example: **Triangular Initial Condition *u(x,0)***

$$u(x,0) = f(x) = \begin{cases} h\frac{2x}{a} & for\ \ 0 < x < \frac{L}{2} \\ h\left(2 - \frac{2x}{L}\right) & for\ \ \frac{L}{2} < x < L \end{cases}$$

$$\frac{\partial u(x,0)}{\partial t} = g(x) = 0 \rightarrow B_n = 0$$




$$A_n = \frac{2}{L}\int_0^{L/2} h\frac{2x}{a}\sin\left(\frac{n\pi}{L}x\right)dx + \frac{2}{L}\int_{L/2}^{L} h\left(2 - \frac{2x}{L}\right)\sin\left(\frac{n\pi}{L}x\right)dx$$

$$A_n = \frac{8h}{\pi^2}\left[\sin\left(\frac{n\pi}{2}\right)/n^2\right]$$



Using: $\sin(A)\cos(B) = \frac{1}{2}[\sin(A+B) + \sin(A-B)]$

$$u(x,t) = \frac{1}{2}\sum_{n=1}^{\infty}\left[A_n \sin\left(\frac{n\pi}{L}(x-ct)\right) + A_n \sin\left(\frac{n\pi}{L}(x+ct)\right)\right]$$

Two travelling waves in opposite directions: Resulting in Standing Waves