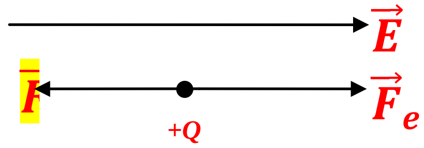
CHAPTER (4)

ENERGY AND POTENTIAL

Work done in moving point charge:

If a + ve point charge Q is related in $\vec{E}$, it will gain kinetic energy (K.E) from the electric field.



$$\vec{F}_e = Q\vec{E}$$

$$\vec{F}_a = -Q\vec{E}$$

The energy taken from the electric field $\bar{E}$ is:

$$dW_e = \vec{F}_e . \vec{dl} = Q\vec{E} . \vec{dl}$$

Then in order to maintain the energy in equilibrium position a force $\vec{F}_a$ must be applied, the work done due to the force $\vec{F}_a$ is:

$$dW_a = \vec{F}_a . \vec{dl} = -Q\vec{E} . \vec{dl}$$

$$w_a = \int dW_a = \int \vec{F}_a . \vec{dl} = -Q \int \vec{E} . \vec{dl} \quad Joule$$

Generally: the work done to move charge Q from point b to point a is:

$$w_a = -Q \int_b^a \vec{E} . \vec{dl} \quad Joule$$

Note: the motion is against the electric field.

Remember:

$\vec{dl} = \hat{a}_x dx + \hat{a}_y dy + \hat{a}_z dz$

Cartesian coordinates

$\vec{dl} = \hat{a}_{r_c} dr_c + \hat{a}_\varphi r_c d\varphi + \hat{a}_z dz$

Cylindrical coordinates

$\vec{dl} =$

Spherical Coordinates

Example:

An electric field is given by:

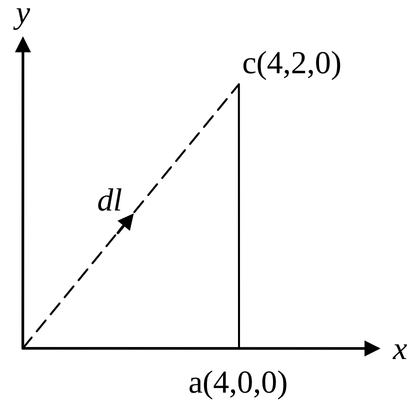
$$\vec{E} = \left(\frac{x}{2} + 2y\right)\hat{a}_x + 2x\hat{a}_y$$

Find the work done in moving point charge $Q = -20\ \mu c$:

i- From point o (0,0,0) to a (4,0,0)
ii- From point a (4,0,0) to c (4,2,0)
iii- From point o (0,0,0) to c (4,2,0)

Solution:

i- From point o (0,0,0) to a (4,0,0)
Through the path from 0 to a
y = 0 and dy = 0



$$w_a = -Q\int_o^a \vec{E}\,.\vec{dl} \quad Joule$$

$$\bar{E}.d\bar{l} = \left[\left(\frac{x}{2} + 2y\right)\hat{x} + 2x\,\hat{y}\right].\left[dx\hat{x} + dy\,\hat{y} + dz\,\hat{z}\right]$$

$$= (\frac{x}{2} + 2y)dx + 2xdy + 0$$

$$\therefore W = -Q\int_0^4 (\frac{x}{2} + 2y)dx + 2xdy\Big|_{dy=0}$$

$$W = -Q\left[\frac{x^2}{4}\right]_0^4 = -\left[-20*10^{-6}\right]\frac{16}{4} = 80\mu J$$

ii- From point a (4,0,0) to c (4,2,0)

Through the path from a to c

x = 4 and dx = 0

$$w_a = -Q\int_a^c \vec{E}.\vec{dl} \quad Joule$$

$$W = -Q\int\left(\frac{x}{2}+2y\right)dx + 2xdy\Big|_{dx=o}^{4}$$

$$= -Q\int_0^2 2xdy\Big|_{x=4} = -8Q(y)_0^2 = -8(-20*10^{-6})2 = 320\mu J$$

iii- From point o (0,0,0) to c (4,2,0)

To go from o (0,0,0) to c (4,2,0) through the path

o → b → c

$$W = (80+320)*10^{-6} = 400\mu J$$

iv- From point o (0,0,0) to c (4,2,0)

To go from o (0,0,0) to c (4,2,0) through the path

o → c

$$w_a = -Q\int_o^c \vec{E}.\vec{dl} \quad Joule$$

$$W = -Q\int \bar{E}.d\bar{l} = -Q\int_{(0,0,0)}^{(4,2,0)}\left(\frac{x}{2}+2y\right)dx + 2xdy$$

Since along the path o → c, both x and y vary, so we have to determine the relation between the two variables by determining the equation of the line ac

$$y = mx + c \qquad , \qquad m = \frac{2}{4} = \frac{1}{2}$$

$$y = \frac{x}{2} + c$$ by taking any two points on the curve

$$2 = \frac{4}{2} + c \qquad \therefore c = 0$$

$$\therefore y = \frac{x}{2} \qquad \rightarrow \qquad dy = \frac{dx}{2}$$

$$\therefore W = -Q\int_0^4 \left[\frac{x}{2} + (\frac{2x}{2}) + (\frac{2x}{2}) \right] dx$$

$$W = -Q\int_0^4 (\frac{5}{2}x)dx = \left[\frac{-Q5}{2} - \frac{x^2}{2} \right]_0^4 = 400\mu J$$

Conclusion:

The work done from $0 \rightarrow a \rightarrow c$ across the path L equal to the work done from $o \rightarrow c$ across the path L', i.e. the work done from $o \rightarrow c$ dose not depend on the path, therefore the electric field in this case called conservative field.

Electric Potential Difference Between Two Points:

The potential difference between two points is the work done needed to move a positive unit charge from one point (b) to the other point (a) against the electric field direction.

$$V_{ab} = V_a - V_b$$

$$V_{ab} \triangleq - \int_b^a \vec{E} \cdot \vec{dl} \qquad (NC^{-1}m, JC^{-1}, V)$$

- **In evaluating line integral, the differential distance $\vec{dl}$ is always positive, even though the path is directed in a decreasing coordinate value.**

- **Also, the lower limit (b) is the initial point, while the upper limit (a) is the final point.**

- **If V_{ab} is positive, point (a) is at a higher potential energy level than point (b).**

The Potential Difference Between Two Points Due to a Point Charge Q

$$V_{ab} \triangleq - \int_b^a \vec{E} \cdot \vec{dl}$$

Where,

$$\vec{E} = \hat{a}_{r_s} \frac{Q}{4\pi\varepsilon r_s^2}$$

$$\vec{dl} =$$

$$V_{ab} = - \int_b^a \vec{E} \cdot \vec{dl} = - \int_b^a \hat{a}_{r_s} \frac{Q}{4\pi\varepsilon r_s^2} \cdot \hat{a}_{r_s} dr_s$$

$$V_{ab} =$$

$$V_{ab} = \frac{Q}{4\pi\varepsilon} \left[\frac{1}{r_{sa}} - \frac{1}{r_{sb}} \right]$$

Note:

If the point (b) is at infinity, so $r_{sb} = \infty$

$$V_{a\infty} = V_a = \frac{Q}{4\pi\varepsilon} \left[\frac{1}{r_{sa}} - \frac{1}{\infty} \right] = \frac{Q}{4\pi\varepsilon r_{sa}}$$

V_a **is called the absolute potential of point a**

Absolute Potential

Is the potential difference with the reference point (b) at infinity.

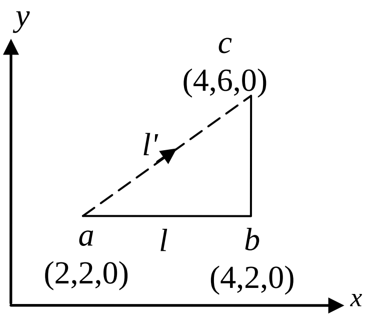
$$V_a = V_{ab}\big|_{b\to\infty} = -\int_{\infty}^{a} \vec{E} \cdot \overrightarrow{dl}$$

Example:

Find V_{ac} by integrating over the paths l, l' as shown in figure. Where,

$$\vec{E} = -\hat{a}_x\, y - \hat{a}_y\, x$$

Solution:



Indirect path a $\rightarrow b \rightarrow c$

$$V_{ac} = -\int_{a}^{b} \vec{E} \cdot \overrightarrow{dl} - \int_{b}^{c} \vec{E} \cdot \overrightarrow{dl}$$

$$\vec{E} = -\hat{a}_x\, y - \hat{a}_y\, x$$

$$\overrightarrow{dl} = \hat{a}_x dx + \hat{a}_y dy + \hat{a}_z dz$$

$$\vec{E}.\vec{dl} = -y\,dx - xdy$$

$$V_{ac} = -\int_{x=2}^{x=4} (-y\,dx - xdy)|_{dy=0,y=2} - \int_{y=2}^{y=6} (-y\,dx - xdy)|_{dx=0,x=4}$$

$$V_{ac} = -\int_{x=2}^{x=4} (-2\,)dx - \int_{y=2}^{y=6} (-4)dy$$

$$V_{ac} = 2(4-2) + 4(6-2) = 20\,volt$$

Direct path (l')

$$V_{ac} = -\int_{(2,2,0)}^{(4,6,0)} (-y\,dx - xdy\,)$$

Since along the path l', x varies as y varies, so we have to find the relation between the two variables.

$$y = mx + c$$

$$y = 2x + c \quad \rightarrow \quad m = \frac{4}{2} = 2$$

$$2 = 2(2) + c \quad \rightarrow \quad \therefore c = -2$$

$$\therefore \mathrm{y} = 2\mathrm{x} - 2 \quad , \quad \mathrm{dy} = 2\mathrm{dx}$$

$$V_{ac} = -\int_2^4 -(+2x-2)dx - 2xdx$$

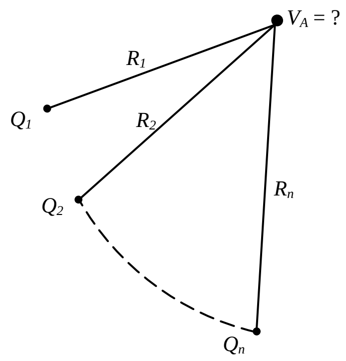
$$= -\int_2^4 (-2x+2-2x)dx = -\int_2^4 (-4x+2)dx$$

$$= \left[-2\frac{x^2}{2} + 2x\right]_2^4 = 20 \text{ volts}$$

$\therefore$ The field is conservative.

Potential at a point due to multi-point charges:

$$V_A = \frac{Q_1}{4\pi\varepsilon_0 R_1} + \frac{Q_2}{4\pi\varepsilon_0 R_2} + - - - + \frac{Q_n}{4\pi\varepsilon_0 R_n}$$



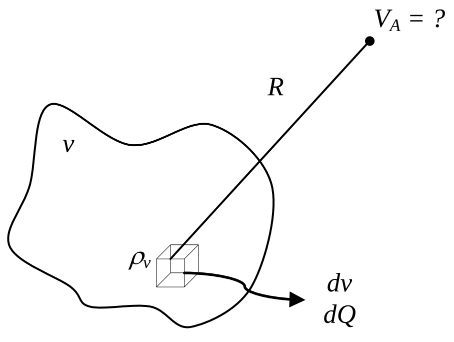
$$V_A = \sum_{i=1}^{n} \frac{Q_i}{4\pi\varepsilon_0 R_i} \quad \text{volt}$$

Potential at a point due to charge distribution

The Building Block for the potential

$$dV_A = \frac{dQ}{4\pi\varepsilon_0 R} = \frac{\rho_v dv}{4\pi\varepsilon_0 R}$$

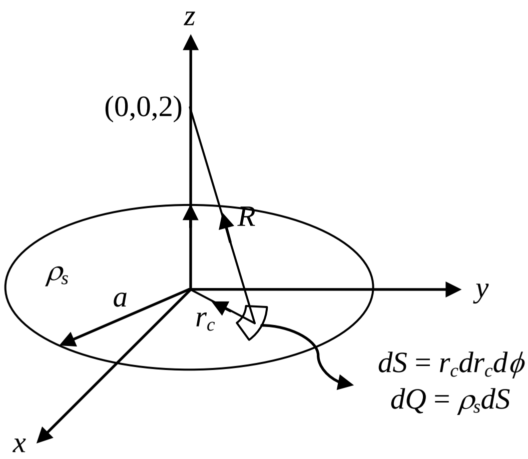
$$V_A = \int \frac{p_v dv}{4\pi\varepsilon_0 R}$$



Example:

A total charge of $\frac{40}{3}$ nc is uniformly distributed in the form of circular disk of radius *a* = 2m. Find the potential at a point on the axes 2m from the disk. Compute this result with that obtained if all charge is concentrated on the center of the disk.

Solution:



$$\rho_s = \frac{Q}{\pi a^2}$$

$$\rho_s = \frac{\frac{40}{3} * 10^{-9}}{\pi (2)^2} = \frac{10^{-8}}{3\pi} \text{ C/m}^2$$

$$dV = \frac{dQ}{4\pi\varepsilon_0 R}$$

$$dV = \frac{\rho_s r_c dr_c d\phi}{4\pi\varepsilon_0 \sqrt{r_c^2 + 4}}$$

$$\therefore V = \frac{\rho_s}{4\pi\varepsilon_0} \int_0^{2\pi} \int_0^{a} \frac{r_c dr_c d\phi}{\sqrt{r_c^2 + 4}} = \frac{\rho_s}{2\varepsilon_0} \int_0^{2} \frac{r_c dr_c}{\sqrt{r_c^2 + 4}}$$

$$= \frac{\rho_s}{2 * 2\varepsilon_0} \int_0^{2} \frac{2 r_c dr_c}{\sqrt{r_c^2 + 4}} = \frac{\rho_s}{4\varepsilon_0} \int_0^{2} \frac{dr_c^2}{\sqrt{r_c^2 + 4}}$$

$$= \frac{10^{-8} * 36\pi}{12\pi^2 * 10^{-2}} - \frac{2\pi^2}{2} \left(\sqrt{r_c^2 + 4}\right|_0^2$$

$$= 120(\sqrt{2} - 1) = 49.7 \quad Volt$$

$$\therefore V_A = \frac{Q}{4\pi\varepsilon_0 R} = \frac{\frac{40}{3} * 10^{-9}}{4\pi\varepsilon_0 2} = 60 \text{ volt}$$

Gradient of potential difference:

$$\text{let}: \quad V = f(x, y, z)$$

$$dV = \frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy + \frac{\partial V}{\partial z}dz \quad \rightarrow \quad (1)$$

$$\nabla V = \frac{\partial V}{\partial x}\hat{x} + \frac{\partial V}{\partial y}\hat{y} + \frac{\partial V}{\partial z}\hat{z}$$

$$\nabla V.d\bar{l} = \left(\frac{\partial V}{\partial x}\hat{x} + \frac{\partial V}{\partial y}\hat{y} + \frac{\partial V}{\partial z}\hat{z}\right).(dx\hat{x} + dy\hat{y} + dz\hat{z})$$

$$\therefore \nabla V.d\bar{l} = \frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy + \frac{\partial V}{\partial z}dz \quad \rightarrow \quad (2)$$

From equations (1) and (2)

$$dV = \nabla V.d\bar{l}$$

$$V = \int \nabla V.d\bar{l} \quad \rightarrow \quad (3)$$

$$V = -\int \bar{E}.d\bar{l} \quad \rightarrow \quad (4)$$

from equations (3), (4) :

$$\vec{E} = -\nabla V$$

The general form of ∇V is:

$$\nabla V = \frac{1}{h_1}\frac{\partial V}{\partial \mu_1}\hat{a}_{\mu_1} + \frac{1}{h_2}\frac{\partial V}{\partial \mu_2}\hat{a}_{\mu_2} + \frac{1}{h_3}\frac{\partial V}{\partial \mu_3}\hat{a}_{\mu_3}$$

Example:

Find the electric field intensity at a point of distance r_s from a point charge Q using $\overrightarrow{E} = -\nabla V$

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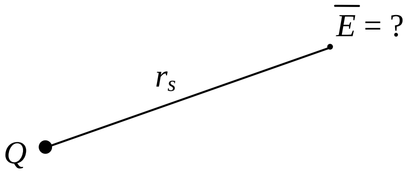
Solution:



$$V = \frac{Q}{4\pi\varepsilon_0 r_s}$$

$$-\nabla V = -\left[\frac{1}{h_1}\frac{\partial v}{\partial r_s}\hat{r}_s + \frac{1}{h_2}\frac{\partial v}{\partial \theta}\hat{\theta} + \frac{1}{h_3}\frac{\partial v}{dQ}\hat{Q}\right]$$

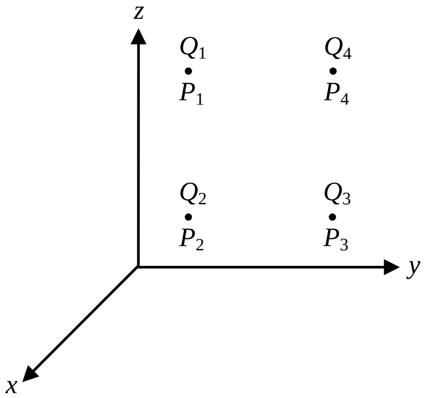
$$\text{Where: } h_1 = 1 \quad , \quad h_2 = \hat{r}_s \quad , \quad h_3 = r_s \sin\theta$$

$$\therefore \overline{E} = -\nabla V = -\left[\frac{\partial}{\partial r_s}\left(\frac{Q}{4\pi\varepsilon_0 r_s}\right)\hat{r}_s\right]$$

$$= -\frac{Q}{4\pi\varepsilon_0}\left(-\frac{1}{r_s^2}\right)\hat{r}_s = \frac{Q}{4\pi\varepsilon_0 r_s^2}\hat{r}_s$$

Energy expended to build up system of charges:

It is required to calculate the energy required to build up a system of 4-point charges $Q_1 \rightarrow Q_4$ as shown in figure.



Procedure:

1- All charges are initially at ∞.

2- We bring $Q_1 \rightarrow Q_2 \rightarrow Q_3 \rightarrow Q_4$. If we bring Q_1 from infinity to point P_1: $\therefore W_1 = 0$

3- If we bring Q_2 from ∞ to point P_2 the work done:

$$\therefore W_2 = Q_2\left[-\int \bar{E}.d\bar{l}\right] = Q_2 V_{21}$$

4- If we bring Q_3 from ∞ to point P_3:

$$\therefore W_3 = Q_3(V_{31} + V_{32})$$

5- If we bring charge Q_4 from ∞ to point P_4:

$$\therefore W_4 = Q_4(V_{41} + V_{42} + V_{43})$$

$$\therefore W_{total} = W_1 + W_2 + W_3 + W_4 = 0 + Q_2 V_{21} + Q_3(V_{31} + +V_{32}) + Q_4(V_{41} + V_{41} + V_{43}) \quad \rightarrow \quad (1)$$

if: $Q_4 \rightarrow Q_3 \rightarrow Q_2 \rightarrow Q_1$

$$\therefore W_4 = 0 \quad , \quad W_3 = Q_3 V_{34} \quad , \quad W_2 = Q_2 V_{23} + Q_2 V_{24} \quad , \quad W_1 = Q_1 V_{13} + Q_1 V_{12} + Q_1 V_{14}$$

$$\therefore W_{total} = 0 + Q_3 V_{34} + Q_2 V_{23} + Q_2 V_{24} + Q_1 V_{13} + Q_1 V_{12} + Q_1 V_{14} \quad \rightarrow \quad (2)$$

from equation (1) + (2):

$$\therefore 2W = Q_1(V_{12} + V_{13} + V_{14}) + Q_2(V_{21} + V_{23} + V_{24}) + Q_3(V_{31} + V_{32} + V_{34}) + Q_4(V_{41} + V_{42} + V_{43})$$
$$= Q_1 V_1 + Q_2 V_2 + Q_3 V_3 + Q_4 V_4$$

$$\therefore 2W = \sum_{I=1}^{I=4} Q_i V_i$$

$$\therefore W = \frac{1}{2} \sum_{i=1}^{i=4} Q_i V_i$$

$$\text{Generally:} \quad W = \frac{1}{2} \sum_{i=1}^{n} Q_i V_i$$

- If the charges are distributed in volume with ρ_v:

$$W_E = \frac{1}{2} \int V \rho_v \, dv \quad , \quad \text{Where: } V \text{ is the potential of the body}$$

- If the charges are distributed on a surface:

$$W_E = \frac{1}{2} \int V \rho_s \, dS$$

- If the charges are distributed on a line:

$$W_E = \frac{1}{2}\int V\rho_l dl$$

- **Energy expended in terms of the electric field**

$$W_E = \frac{1}{2}\int V\rho_v dv \quad \rightarrow \quad (3)$$

$$\because \rho_v = \nabla.\overline{D} \qquad \qquad \therefore W_E = \frac{1}{2}\int V(\nabla.\overline{D})\, dv$$

$$\because \nabla.(V\overline{D}) = V(\nabla.\overline{D}) + \overline{D}.\nabla V$$

$$\therefore V(\nabla.\overline{D}) = \nabla.(V\overline{D}) - \overline{D}.\nabla V \quad \rightarrow \quad (4)$$

Substitute (4) into (3) we get :

$$W_E = \frac{1}{2}\int_V [\nabla.(V\overline{D}) - \overline{D}.\nabla V]\, dv$$

$$\because \overline{E} = -\nabla V \quad , \quad \overline{D} = \varepsilon_o \overline{E}$$

$$\therefore W_E = \frac{1}{2}\int_V \nabla.(V\overline{D})\, dv - \frac{1}{2}\int_V \varepsilon_o \overline{E}.(-\overline{E})\, dv$$

$$= \frac{1}{2}\int_V \nabla.(V\overline{D})\, dv + \frac{1}{2}\int_V \varepsilon_o E^2\, dv$$

Note:

Divergence theorem : $\oint \overline{A}.d\overline{S} = \int_V \nabla.\overline{A}\, dv$

$$\therefore \int_V \nabla.(V\overline{D})\, dv = \oint V\overline{D}.d\overline{S}$$

But : $V \propto \frac{1}{r_s}$, $D \propto \frac{1}{r_s^2}$, $dS \propto r_s^2$

$$\therefore V\overline{D}.d\overline{S} \propto \frac{1}{r_s}$$

This term decreases rapidly with increasing r_s, i.e. this term $\cong 0$:

$$\therefore W_E = \frac{1}{2}\int_V \varepsilon_o E^2 \, dv$$

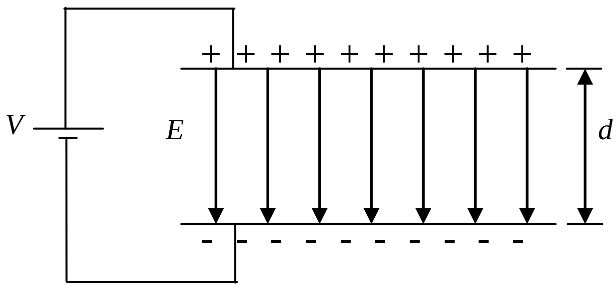
Summary:

$$\mathbf{W_E = \frac{1}{2}\sum_{i=1}^{n} Q_i v_i = \frac{1}{2}\int_V \rho_v V \, dv = \frac{1}{2}\int_V \varepsilon_o E^2 \, dv}$$

Example:

A parallel plate capacitor for which: $C = \frac{\varepsilon_o A}{d}$, has potential V across its plates. Find the energy stored in the electric field.

Solution:



$$\because V = \int \overline{E}.d\overline{l}$$

$$V = -E\int_d^o dl = Ed$$

$$W_E = \frac{1}{2}\int_V \varepsilon_o E^2 \, dv = \frac{1}{2}\int_V \varepsilon_o \left(\frac{V}{d}\right)^2 dv$$

$$= \frac{1}{2}\frac{\varepsilon_o V^2}{d^2}.A d = \frac{1}{2}\frac{\varepsilon_o A}{d} V^2 = \frac{1}{2} C V^2$$

Example:

For a sphere of charge ρ_v and radius a. Find the energy expended to build this system of charges using two formulas.

Solution:

$$W_E = \frac{1}{2}\int_V \rho_V V\,dv \qquad = \frac{1}{2}\int_V \varepsilon_o E^2\,dv$$

$$V = -\int_a^{r_S} \overline{E_1}.d\bar{l} - \int_\infty^a \overline{E_2}.d\bar{l}$$

$$\because \oint \overline{D}.d\overline{S} = Q, \qquad \therefore 4\pi r_s^2.D = \rho_V.\frac{4}{3}\pi r_s^3$$

$$\therefore \overline{\mathbf{D}} = \frac{\rho_V \mathbf{r}_s}{3}\hat{\mathbf{r}}_s$$

$$\therefore \overline{\mathbf{E}_1} = \frac{\rho_V \mathbf{r}_s}{3\varepsilon_o}\hat{\mathbf{r}}_s$$

$$\because Q = \rho_V.\frac{4}{3}\pi a^3$$

$$\therefore \rho_V = \frac{3Q}{4\pi a^3}$$

$$\therefore \overline{\mathbf{E}_1} = \frac{3Q\mathbf{r}_s}{4\pi a^3.3\varepsilon_o}\hat{\mathbf{r}}_s = \frac{Q\mathbf{r}_s}{4\pi\varepsilon_o a^3}\hat{\mathbf{r}}_s$$

$$\because \mathbf{D}_2.4\pi \mathbf{r}_s^2 = \mathbf{Q}$$

$$\therefore \overline{\mathbf{E}_2} = \frac{\mathbf{Q}}{4\pi\varepsilon_o \mathbf{r}_s^2}\hat{\mathbf{r}}_s$$

$$\therefore V = -\int_a^{r_s} \frac{Qr_s}{4\pi\varepsilon_o a^3}\, dr_s - \int_\infty^a \frac{Q}{4\pi\varepsilon_o r_s^2}\, dr_s$$

$$= -\frac{Q}{4\pi\varepsilon_o a^3}\left(\frac{r_s^2}{2}\right)_a^{r_s} - \frac{Q}{4\pi\varepsilon_o}\left(-\frac{1}{r_s}\right)_\infty^a$$

$$\therefore V = -\frac{Q}{8\pi\varepsilon_o a^3}(r_s^2 - a^2) + \frac{Q}{4\pi\varepsilon_o a}$$

$$\therefore W_E = \frac{1}{2}\int_V \rho_V V\, dv$$

$$= \frac{1}{2}\int_0^{2\pi}\int_0^{\pi}\int_0^{a}\left(\frac{3Q}{4\pi a^3}\right)\left(\frac{Q}{4\pi\varepsilon_o a} - \frac{Qr_s^2}{8\pi\varepsilon_o a^3} + \frac{Q}{8\pi\varepsilon_o a}\right) r_s^2 \sin\theta\, dr_s d\theta\, d\phi$$

$$= \frac{1}{2}\cdot\frac{3Q}{4\pi a^3}\cdot\frac{Q}{8\pi\varepsilon_o a}(4\pi).\int_0^a\left[(2+1) - \frac{r_s^2}{a^2}\right] r_s^2\, dr_s$$

$$= \frac{3Q^2}{16\pi\varepsilon_o a^4}\left(r_s^3 - \frac{r_s^5}{5a^2}\right)_0^a$$

$$\therefore W_E = \frac{3Q^2}{16\pi\varepsilon_o a^4}\left(a^3 - \frac{a^3}{5}\right) = \frac{3Q^2}{16\pi\varepsilon_o a^4}\left(\frac{4}{5}a^3\right) = \frac{3Q^2}{20\pi\varepsilon_o a}$$

Another method:

$$W_E = \frac{1}{2}\int_V \varepsilon_o E^2\, dv$$

$$= \frac{1}{2}\int \varepsilon_o E_2^2\, dv + \frac{1}{2}\int \varepsilon_o E_1^2\, dv$$

$$\therefore W_E = \frac{1}{2}\int_0^{2\pi}\int_0^{\pi}\int_a^{\infty}\varepsilon_o\left(\frac{Q}{4\pi\varepsilon_o r_s^2}\right)^2 r_s^2 \sin\theta\, dr_s d\theta\, d\varphi + \frac{1}{2}\int_0^{2\pi}\int_0^{\pi}\int_0^{a}\varepsilon_o\left(\frac{Qr_s}{4\pi\varepsilon_o a^3}\right)^2 r_s^2 \sin\theta\, dr_s d\theta\, d\phi$$

$$= \frac{1}{2}.4\pi\varepsilon_o\left(\frac{Q}{4\pi\varepsilon_o}\right)^2\int_\infty^a \frac{1}{r_s^2}\, dr_s + \frac{1}{2}.4\pi\varepsilon_o\left(\frac{Q}{4\pi\varepsilon_o a^3}\right)^2\int_0^a r_s^4\, dr_s$$

$$= \frac{2\pi\varepsilon_o Q^2}{16\pi^2\varepsilon_o^2}\left(-\frac{1}{r_s}\right)_a^\infty + \frac{2\pi\varepsilon_o Q^2}{16\pi^2\varepsilon_o^2 a^6}\left(\frac{r_s^5}{5}\right)_0^a$$