Research Article

Mathematical Methods in the Applied Sciences

Received 27 January 2012

Published online 28 June 2013 in Wiley Online Library

(wileyonlinelibrary.com) DOI: 10.1002/mma.2802
MOS subject classification: 22E05; 35Q53; 54H15

Lie-group method for solving the problem of fission product behavior in nuclear fuel

Mina B. Abd-el-Malek[a*†] and Hossam S. Hassan[b]

Communicated by J. E. Muñoz Rivera

Calculation of the gas atom concentration is an important feature of all physical models of fission gas release. We apply Lie-group method for determining symmetry reductions to the diffusion equation describing the fission gas release from nuclear fuel. The resulting nonlinear ordinary differential equation is solved numerically using nonlinear finite difference method. Effects of the dimensionless group constant, the time, and the grain radius on the concentration diffusion function have been studied, and the results are plotted. It is found that the concentration of gas atoms increases as the dimensionless group constant, the power index, and the time increase, and it decreases with increase of the grain radius. Copyright © 2013 John Wiley & Sons, Ltd.

Keywords: diffusion equation; finite difference; similarity solutions; Lie group

1. Introduction

The problem of fission gas release from nuclear fuel has been fully studied by many investigators. The majority of the previous analytical or experimental works on fission gas release has been primarily focussed on the gas released with the change in temperature.

Hayns and Wood [1] have used two separate models to discuss gas release in a fast reactor oxide fuel element. One model was based on the random motion and coalescence of gas atoms and bubbles within fuel grains. The second model was based on the biased migration and coalescence of gas bubbles in a thermal gradient. They have demonstrated that the models do not provide a unique description of the gas release, especially at low temperatures.

Mac Innes and Brearley [2] studied a model for the release of fission gas from reactor fuel undergoing transient heating. They have incorporated this model into a description of the evolution of fission gas from irradiated fuel heated rapidly to melting at 150 K/s. They have demonstrated that the high gas releases observed in this type of experiment are due to thermal resolution from bubbles together with single gas atom diffusion to the grain boundaries.

Matthews and Wood [3] suggested an approach to solving the problem of diffusion flow to a spherical boundary, which uses an approximation describing the intragranular concentrations of the diffusing species in terms of quadratic functions in two concentric regions. They have used a variational principle to calculate the radial distribution of the concentration.

Dowling *et al*. [4] studied the effect of irradiation-induced resolution of fission gas release. They concluded that the numerical calculations were insensitive to the resolution model chosen except at low temperatures where, because gas atom diffusion is very slow, release is strongly controlled by resolution. Under such low-temperature conditions, the grain-boundary concentration increases linearly with time.

Rest [5] presented an improved model for fission product behavior in nuclear fuel under normal and accident conditions.

Abd-el-Malek *et al*. [6] presented a general procedure for applying a one-parameter group transformation to the diffusion equation describing fission gas release from nuclear fuel. Under the transformation, the partial differential equation with boundary conditions is reduced to an ordinary differential equation with the appropriate corresponding conditions. The equation is then solved numerically to calculate approximated value of concentration diffusion function.

In 1992, Sophocleous [7] applied the symmetries for studying radially nonlinear symmetric diffusion equations. In 2007, Vaneeva *et al*. [8] applied the group analysis method to solve problems of variable coefficient reaction diffusion equations. Recently, Vaneeva *et al*. [9] in 2009 applied the group analysis method and found exact solutions of variable coefficient semilinear equations with a power source.

[a] *Department of Engineering Mathematics and Physics, Faculty of Engineering, Alexandria University, Alexandria 21544, Egypt*
[b] *Department of Basic and Applied Science, Arab Academy for Science, Technology and Maritime Transport, P.O.BOX 1029 Alexandria, Egypt*
**Correspondence to: Mina B. Abd-el-Malek, Department of Engineering Mathematics and Physics, Faculty of Engineering, Alexandria University, Alexandria 21544, Egypt.*
† *E-mail: minab@aucegypt.edu*

In this work, Lie-group method is applied to the diffusion equation describing the fission gas release from nuclear fuel, for determining symmetry reductions of the given partial differential equation, [10–22]. The resulting nonlinear ordinary differential equation is solved numerically using nonlinear finite difference method, and the results are plotted.

We could not get the potential symmetries of the diffusion equation describing the fission gas release from nuclear fuel because it cannot write in conservation form. Pucci and Saccomandi [23] determined the necessary conditions for a partial differential equation of order n in the unknown function $u(x,t)$ to be written in the conservation form. They stated that the partial differential equation must be in the form

$$\mathrm{D}_t \mathrm{A} - \mathrm{D}_x \mathrm{B} = 0, \tag{1.1}$$

where A and B are functions in $(x, t, u, u_1, u_2, ...u_{n-1})$, u_k stands for the set of kth order derivatives of $u(x,t)$, whereas D_t and D_x are the operators of the total derivatives with respect to t and x . Clearly, the partial differential equation can be written in the conservative form (1.1) only if it is quasi-linear.

2. Mathematical formulation of the problem

It is necessary to calculate the loss of gas to the grain boundaries by diffusion of single gas atoms and gas bubbles in the modeling of fission gas behavior in nuclear fuel. To simplify the problem, we assume that grains within the fuel are spherical [5]. If the only sink for gas atoms is the boundary itself, the concentration of gas atoms within the spherical grain satisfy the equation

$$\frac{\partial C_1}{\partial t} = \frac{1}{r^2}\frac{\partial}{\partial r}\left[\mathrm{D}(C_1)\, r^2 \frac{\partial C_1}{\partial r}\right] + F \quad , \quad 0 < r < d, \quad 0 < t < T, \tag{2.1}$$

together with the following initial and boundary conditions

$$\begin{aligned} &\text{(i) } C_1(r,0) = 0,\ 0 \le r \le d,\\ &\text{(ii) } C_1(d,t) = 0,\ 0 \le t \le T,\\ &\text{(iii) } \frac{\partial C_1}{\partial r}(0,t) = 0,\ 0 \le t \le T, \end{aligned} \tag{2.2}$$

where $C_1(r,t)$ is the concentration of gas atoms, $\mathrm{D}(C_1)$ is the gas atoms diffusion coefficient, F is the rate of generation of the gas atoms, d is the grain radius, and T is the time interval.

Assuming that

$$\mathrm{D}(C_1) = \mathrm{D}_0 C_1^n, \tag{2.3}$$

where D_0 and n are constants.

The variables in (2.1) are dimensionless according to

$$Z = \frac{r}{d}, \tau = \frac{t\,\mathrm{D}_0}{d^2}, \quad C = \frac{C_1}{C_0} \quad , \quad \mathrm{D}(C_1) = \mathrm{D}_0 C^{\mathrm{n}}, \tag{2.4}$$

where C_0 is the maximum concentration of the gas atoms at the center.

Substitution of (2.3) and (2.4) into (2.1) yields

$$\frac{\partial C}{\partial \tau} - \frac{2\,C^n}{Z}\frac{\partial C}{\partial Z} - n\,C^{n-1}\left[\frac{\partial C}{\partial Z}\right]^2 - C^n \frac{\partial^2 C}{\partial Z^2} - AF = 0, \tag{2.5}$$

where, $\mathrm{A} = \frac{d^2}{C_0\,\mathrm{D}_0}$ is a constant.

The initial and boundary conditions (2.2) will be

$$\begin{aligned} &\text{(i) } C(Z,0) = 0,\ 0 \le Z \le 1,\\ &\text{(ii) } C(1,\tau) = 0,\ 0 \le \tau \le H,\\ &\text{(iii) } \frac{\partial C}{\partial Z}(0,\tau) = 0,\ 0 \le \tau \le H, \end{aligned} \tag{2.6}$$

where $\mathrm{H} = \frac{T\,\mathrm{D}_0}{d^2}$ is a constant.

3. Solution of the problem

At first, we derive the similarity solutions by using Lie-group method under which (2.5) and the initial and boundary conditions (2.6) are invariant, and then we use these symmetries to determine the similarity variables.

3.1. *Lie point symmetries*

Consider the one-parameter (ε) Lie group of infinitesimal transformations in $(Z,\tau;C,F)$ given by

$$\bar{Z} = Z + \varepsilon\,\phi\,(Z,\tau;C,F) + O(\varepsilon^2),$$

$$\bar{\tau} = \tau + \varepsilon\,\zeta(Z,\tau;C,F) + O(\varepsilon^2),$$

$$\bar{C} = C + \varepsilon\,\eta\,(Z,\tau;C,F) + O(\varepsilon^2),$$

$$\bar{F} = F + \varepsilon\,\Psi\,(Z,\tau;C,F) + O(\varepsilon^2), \tag{3.1}$$

where 'ε' is a small parameter.

The partial differential equation (2.5) is said to admit a symmetry generated by the vector field

$$\boldsymbol{X} \equiv \phi\frac{\partial}{\partial Z} + \zeta\frac{\partial}{\partial\tau} + \eta\frac{\partial}{\partial C} + \Psi\frac{\partial}{\partial F}, \tag{3.2}$$

if it is left invariant by the transformation $(Z,\tau;C,F) \to \left(\bar{Z},\bar{\tau};\bar{C},\bar{F}\right)$.

The solutions $C = C(Z,\tau)$ and $F = F(Z,\tau)$, are invariant under the symmetry (3.2) if

$$\Phi_C \equiv X\,(C - C(Z,\tau)) = 0, \text{ when } C = C(Z,\tau), \tag{3.3}$$

and

$$\Phi_F \equiv X\,(F - F(Z,\tau)) = 0, \text{ when } F = F(Z,\tau). \tag{3.4}$$

Write

$$\Delta \equiv C_\tau - \frac{2\,C^n}{Z}C_Z - n\,C^{n-1}\,[C_Z]^2 \; - C^n C_{ZZ} - AF = 0, \tag{3.5}$$

where subscripts denote partial derivatives.

A vector $\boldsymbol{X}$ given by (3.2) is said to be a Lie-point symmetry vector field for (2.5) if

$$\boldsymbol{X}^{[2]}\ (\Delta)\,|_{\Delta=0} = 0, \tag{3.6}$$

where

$$\boldsymbol{X}^{[2]} \equiv \phi\frac{\partial}{\partial Z} + \zeta\frac{\partial}{\partial\tau} + \eta\frac{\partial}{\partial C} + \Psi\frac{\partial}{\partial F} + \eta^Z\frac{\partial}{\partial C_Z} + \eta^\tau\frac{\partial}{\partial C_\tau} + \eta^{ZZ}\frac{\partial}{\partial C_{ZZ}} \tag{3.7}$$

is the second prolongation of $\boldsymbol{X}$.

To calculate the prolongation of a given transformation, we need to differentiate (3.1) with respect to each of the parameters, Z and τ. To do this, we introduce the following total derivatives:

$$\mathrm{D}_Z \equiv \partial_Z + C_Z\,\partial_C + F_Z\,\partial_F + C_{ZZ}\partial_{C_Z} + C_{Z\tau}\partial_{C_\tau} + \ldots\ldots,$$

$$\mathrm{D}_\tau \equiv \partial_\tau + C_\tau\,\partial_C + F_\tau\,\partial_F + C_{Z\tau}\partial_{C_Z} + C_{\tau\tau}\partial_{C_\tau} + \ldots \tag{3.8}$$

Equation (3.6) gives the following partial differential equation:

$$\frac{2\,C^n\,\phi}{Z^2}\,C_Z - \left[\frac{2\,n\,C^{n-1}}{Z}\,C_Z + n\,(n-1)\,C^{n-2}[C_Z]^2 + n\,C^{n-1}\,C_{ZZ}\right]\eta - A\,\Psi$$

$$+\,\eta^\tau - \left[\frac{2\,C^n}{Z} + 2n\,C^{n-1}\,C_Z\right]\eta^Z - C^n\,\eta^{ZZ} = 0. \tag{3.9}$$

The components $\eta^Z, \eta^\tau, \eta^{ZZ}$ can be determined from the following expressions:

$$\begin{aligned} \eta^Z &= \mathrm{D}_Z\,\eta - C_Z\mathrm{D}_Z\,\phi - C_\tau\mathrm{D}_Z\,\zeta,\\ \eta^\tau &= \mathrm{D}_\tau\,\eta - C_Z\mathrm{D}_\tau\,\phi - C_\tau\mathrm{D}_\tau\,\zeta,\\ \eta^{ZZ} &= \mathrm{D}_Z\,\eta^Z - C_{ZZ}\mathrm{D}_Z\,\phi - C_{Z\tau}\mathrm{D}_Z\,\zeta. \end{aligned} \tag{3.10}$$

Substitution of (3.10) into (3.9) will lead to a large expression, then equating to 0 the coefficients of $C_{Z\tau}$, $C_Z C_{Z\tau}$, $F_\tau C_{Z\tau}$, $C_Z C_\tau$, and $F_\tau C_Z$ gives

$$\zeta_Z = \zeta_C = \zeta_F = \phi_C = \phi_F. \tag{3.11}$$

Substitution of (3.11) into (3.9) will remove many terms. Then, equating to 0 the coefficients of C_Z, C_τ, $(C_Z)^2$, F_τ, and the remaining terms leads to the following system of determining equations:

$$\begin{aligned}
&\frac{2C^n}{Z^2}\phi - \phi_\tau - 2nC^{n-1}\eta_Z - \frac{2C^n}{Z}\phi_Z - 2C^n\eta_{ZC} + C^n\phi_{ZZ} = 0,\\
&\eta = \frac{C}{n}(2\phi_Z - \zeta_\tau),\\
&n\,C^{n-2}\eta - n\,C^{n-1}\eta_C - \ C^n\eta_{CC} = 0,\\
&\eta_F = 0,\\
&\eta_\tau - A\,\Psi - \frac{2C^n}{Z}\eta_Z - C^n\eta_{ZZ} + \frac{nA\eta F}{C} + AF\eta_C - 2AF\phi_Z = 0.
\end{aligned} \tag{3.12}$$

Solving the system of determining equations (3.12), in view of the invariance of the initial and boundary conditions (2.6), yields

$$\phi = K_1 Z,$$

$$\zeta = (2K_1 - nK_2)\,\tau + K_3,$$

$$\eta = K_2 C,$$

$$\psi = [(n+1)K_2 - 2K_1]\,F. \tag{3.13}$$

The nonlinear equation (2.5) has the three-parameter Lie group of point symmetries generated by

$$X_1 \equiv \frac{\partial}{\partial\tau} \quad , \quad X_2 \equiv Z\frac{\partial}{\partial Z} + 2\tau\frac{\partial}{\partial\tau} - 2F\frac{\partial}{\partial F}, \quad X_3 \equiv -n\tau\frac{\partial}{\partial\tau} + C\frac{\partial}{\partial C} + (n+1)F\frac{\partial}{\partial F}. \tag{3.14}$$

The one-parameter group generated by X_1 consists of translation, whereas the remaining symmetries X_2 and X_3 generate scaling.

The finite transformations corresponding to the symmetries X_1, X_2, and X_3 are respectively

$$\begin{aligned}
&X_1: \overline{Z} = Z, &&\overline{\tau} = \tau + \varepsilon_1, &&\overline{C} = C, &&\overline{F} = F,\\
&X_2: \overline{Z} = e^{\varepsilon_2}Z, &&\overline{\tau} = e^{2\varepsilon_2}\tau, &&\overline{C} = C, &&\overline{F} = e^{-2\varepsilon_2}F,\\
&X_3: \overline{Z} = Z, &&\overline{\tau} = e^{-n\varepsilon_3}\tau, &&\overline{C} = e^{\varepsilon_3}C, &&\overline{F} = e^{(n+1)\varepsilon_3}F,
\end{aligned} \tag{3.15}$$

where ε_1, ε_2, and ε_3 are the group parameters.

Table I illustrates the solutions of the invariant surface conditions for some operators of the three-parameter Lie group of point symmetries.

As seen from Table I, the solution of the invariant surface condition (3.3) under X_1 is $C \equiv C(Z)$, which contradicts the boundary conditions.

The solutions of the invariant surface conditions under X_2 are $C \equiv C\left(\frac{Z}{\sqrt{\tau}}\right)$ and $F \equiv \frac{1}{\tau}F\left(\frac{Z}{\sqrt{\tau}}\right)$, which are solutions of (2.5), even though they are not particularly interesting because they contradict the boundary conditions.

On the other hand, the solutions of the invariant surface conditions X_3 are

$$C = \tau^{-\frac{1}{n}}g_1(\alpha), \quad F = \tau^{-1-\frac{1}{n}}g_2(\alpha), \tag{3.16}$$

where $\alpha = Z$ is the similarity variable.

Substitution of (3.16) into (2.5) yields

$$(g_1)^n\frac{d^2g_1}{d\alpha^2} + \frac{2}{\alpha}(g_1)^n\frac{dg_1}{d\alpha} + n(g_1)^{n-1}\left[\frac{dg_1}{d\alpha}\right]^2 + \frac{1}{n}g_1 + Ag_2 = 0. \tag{3.17}$$

The last term of (3.17), which is related to generation, is called dimensionless group constant (say K). This generation term is considered constant during steady state operation of the nuclear reactor because fission rate is constant. The fission rate is kept constant to assure the continuous constant power release from the reactor. The change in gaseous fission release is only considered on the long run over

Table I. Solutions of the invariant surface conditions.

Generator	Characteristic $\Phi = (\Phi_C; \Phi_F)$	Solution of the invariant surface conditions
X_1	$\Phi_C = -C_\tau \quad , \quad \Phi_F = -F_\tau$	$C \equiv C(Z), \quad F(Z)$
X_2	$\Phi_C = -Z\,C_Z - 2\,\tau\,C_\tau \quad \Phi_F = -2\,F - Z\,F_Z - 2\,\tau\,F_\tau$	$C \equiv C\left(\frac{Z}{\sqrt{\tau}}\right), F \equiv \frac{1}{\tau}F\left(\frac{Z}{\sqrt{\tau}}\right)$
X_3	$\Phi_C = C + n\,\tau\,C_\tau, \quad \Phi_F = (n+1)\,F + n\,\tau\,F_\tau$	$C \equiv \tau^{-\frac{1}{n}} g_1(\alpha)$,
		$F \equiv \tau^{-1-\frac{1}{n}} g_2(\alpha)$, where $\alpha = Z$
$X_1 + \beta\,X_2$	$\Phi_C = -\beta\,Z\,C_Z - (2\,\beta\,\tau + 1)\,C_\tau$	$C \equiv C(\lambda), \quad F \equiv \frac{1}{(2\,\beta\,\tau+1)}F(\lambda)$
	$\Phi_F = -2\,\beta\,F - \beta\,Z\,F_Z - (2\,\beta\,\tau + 1)\,F_\tau$	where $\lambda \equiv \frac{Z}{\sqrt{2\,\beta\,\tau+1}}$
$X_1 + \beta\,X_3$	$\Phi_C = \beta C - (1 - n\,\beta\tau)\,C_\tau$	$C \equiv (1 - n\,\beta\,\tau)^{-\frac{1}{n}} C(Z)$
	$\Phi_F = (1+n)\,\beta F - (1 - n\,\beta\tau)\,F_\tau$	$F \equiv (1 - n\,\beta\,\tau)^{-1-\frac{1}{n}} F(Z)$
$X_2 + \beta\,X_3$	$\Phi_C = \beta C - Z\,C_Z - (2 - n\,\beta)\,\tau\,C_\tau$	$C \equiv (\tau)^{\frac{\beta}{2-n\,\beta}} C(\lambda_1)$
	$\Phi_F = ((n+1)\beta - 2)\,F - Z\,F_Z - (2 - n\,\beta)\,\tau\,F_\tau$	$F \equiv (\tau)^{-1-\frac{\beta}{2-n\,\beta}} F(\lambda_1)$
		where $\lambda_1 \equiv Z\,\tau^{-\frac{1}{2-n\,\beta}}$
$X_1 + \beta\,X_2$	$\Phi_C = \gamma\,C - \beta\,Z\,C_Z - (1 + (2\beta - n\,\gamma)\tau)\,C_\tau$	$C \equiv (1 + (2\,\beta - n\,\gamma)\,\tau)^{\frac{\gamma}{2\beta - n\gamma}}\,C(\lambda_2)$
$+\gamma\,X_3$	$\Phi_F = ((1+n)\gamma - 2\,\beta)\,F - \beta\,Z\,F_Z - (1 + (2\beta - n\,\gamma)\tau)\,F_\tau$	$F \equiv (1 + (2\,\beta - n\,\gamma)\,\tau)^{-1-\frac{\gamma}{2\beta - n\gamma}}\,F(\lambda_2)$
		where, $\lambda_2 \equiv Z\,(1 + (2\,\beta - n\,\gamma)\,\tau)^{-\frac{\beta}{2\beta - n\gamma}}$

the lifetime of fission fuel in the reactor. Because 'F' is the total generation per unit volume and the mass flux is equal to $\frac{D\,C_0}{d}$ (mass flow per unit area), then the last term in (3.17) can be written as

$$\frac{d^2 F\,\frac{4}{3}\,\pi\,d}{\frac{C_0}{d}\,D_0\,\frac{4}{3}\,\pi\,d^2} \equiv \text{total gas generation / leakage } = K.$$

Hence, (3.17) will be

$$(g_1)^n \frac{d^2 g_1}{d\,\alpha^2} + \frac{2}{\alpha}(g_1)^n \frac{d\,g_1}{d\,\alpha} + n\,(g_1)^{n-1}\left[\frac{d\,g_1}{d\,\alpha}\right]^2 + \frac{1}{n}\,g_1 + K = 0. \tag{3.18}$$

For (3.16),for the initial and boundary conditions (2.6) to be satisfied, the value of index 'n' must be negative. Under the similarity variable 'α', the initial and boundary conditions (2.6) will be

$$\text{(i)} g_1\,(1) = 0, \quad \text{(ii)} \frac{d\,g_1(0)}{d\,\alpha} = 0. \tag{3.19}$$

The second order differential equation (3.18) and the conditions (3.19) are the same obtained by Abd-el-Malek *et al.* [6] when $n = -1$.

The general solutions of the invariant surface conditions under $X_1 + \beta\,X_2$ are

$$C \equiv C(\lambda), \quad F \equiv \frac{1}{(2\,\beta\,\tau + 1)} F(\lambda), \tag{3.20}$$

where $\lambda \equiv \frac{Z}{\sqrt{2\,\beta\,\tau+1}}$ is the similarity variable, which does not satisfy the initial condition (2.6i).

For $X_1 + \beta\,X_3$, the general solutions of the invariant surface conditions are

$$C \equiv (1 - n\,\beta\,\tau)^{-\frac{1}{n}} C(Z), \quad F \equiv (1 - n\,\beta\,\tau)^{-1-\frac{1}{n}} F(Z). \tag{3.21}$$

Equation (3.21) is a solution of (2.5), even though it is not a particularly interesting one because it contradicts the initial condition (2.6i).

For $X_2 + \beta\,X_3$, the general solutions of the invariant surface conditions are

$$C \equiv (\tau)^{\frac{\beta}{2-n\,\beta}} C(\lambda_1), \quad F \equiv (\tau)^{-1-\frac{\beta}{2-n\,\beta}} F(\lambda_1), \tag{3.22}$$

where $\lambda_1 \equiv Z\,\tau^{-\frac{1}{2-n\,\beta}}$ is the similarity variable, which does not satisfy the initial condition (2.6i).

For $X_1 + \beta\,X_2 + \gamma\,X_3$, the general solutions of the invariant surface conditions are

$$C \equiv (1 + (2\,\beta - n\,\gamma)\,\tau)^{\frac{\gamma}{2\beta - n\gamma}}\,C(\lambda_2), \quad F \equiv (1 + (2\,\beta - n\,\gamma)\,\tau)^{-1-\frac{\gamma}{2\beta - n\gamma}}\,F(\lambda_2), \tag{3.23}$$

where $\lambda_2 \equiv Z\,\,(1 + (2\,\beta - n\,\gamma)\,\tau)^{-\frac{\beta}{2\beta - n\gamma}}$ is the similarity variable, which does not satisfy the initial condition (2.6i).

4. Numerical solution

The ordinary differential equation (3.18) with the appropriate corresponding conditions (3.19) is solved numerically using nonlinear finite difference method applied to the nonlinear second order boundary value problem.

4.1. Study on the effect of the dimensionless group constant K

Figure 1 illustrates the behavior of the concentration of gas atoms $C(Z,\tau)$ for different values of the dimensionless group constant K at $n=-1$ and $\tau=4$. As seen, the concentration increases as the constant K increases. The dimensionless group constant K is usually less than or equal to unity. If it is higher than unity, this means that the generation exceeds leakage, which accumulates the gas inside the grain, forming a pressure on the grain, leading to fractures in the grain and leakage of the gas.

4.2. Study on the effect of the time τ

Figure 2 illustrates the behavior of the concentration of gas atoms $C(Z,\tau)$ for different values of time τ at $n=-1$ and $K=1.0$. As seen, the concentration increases as the time increases.

4.3. Study on the effect of the grain radius

Figure 3 illustrates the behavior of the concentration of gas atoms $C(Z,\tau)$ for different values of the grain radius at $n=-1$ and $K=0.01$. As seen, the concentration decreases as the grain radius increases.

4.4. Study on the effect of the index n

Figure 4 illustrates the behavior of the concentration of gas atoms $C(Z,\tau)$ for different values of the index n at $\tau=4$ and $K=0.01$. As seen, the concentration increases as the index n increases.

Figure 5 illustrates the maximum concentration of gas atoms $C(Z,\tau)$ for different values of the index n at $\tau=4$ and $K=0.01$.

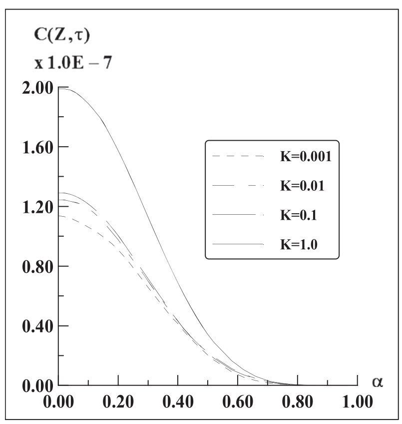

Figure 1. Concentration of gas atoms profiles for different values of the group constant K at $n=-1$ and $\tau=4$.

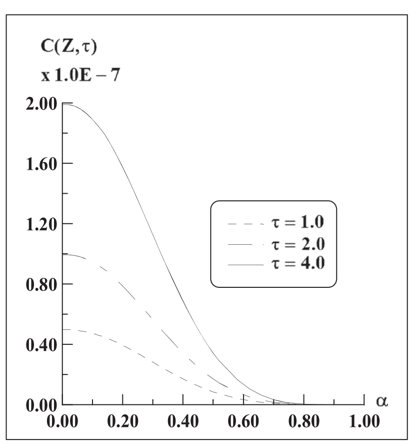

Figure 2. Concentration of gas atoms profiles for different values of time τ at $n=-1$ and $K=1.0$.

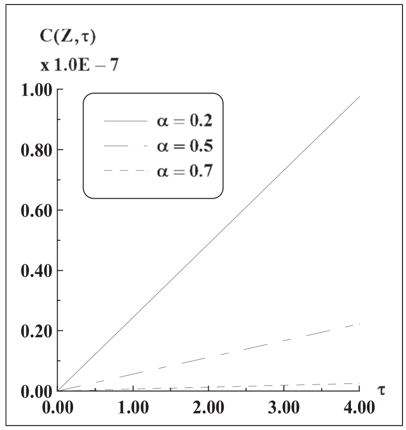

Figure 3. Concentration of gas atoms profiles for different grain radius at $n = -1$ and $K = 0.01$.

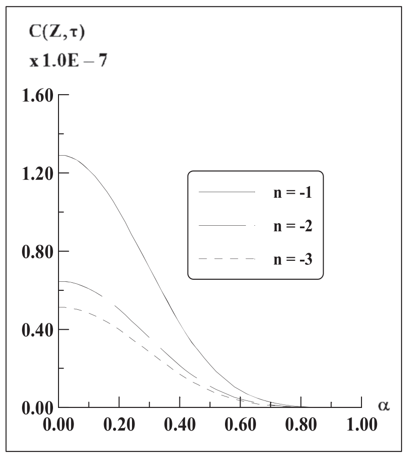

Figure 4. Concentration of gas atoms profiles for different values of 'n' at $\tau = 4$ and $K = 0.01$.

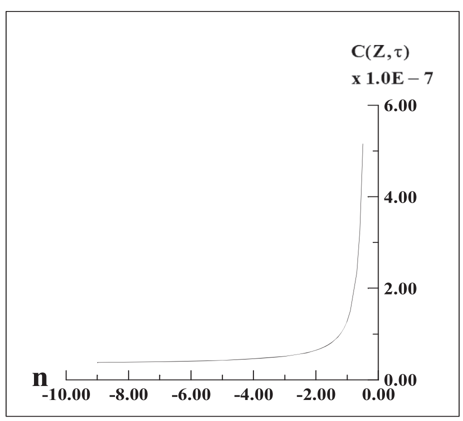

Figure 5. Maximum concentration of gas atoms profiles for different values of 'n' at $\tau = 4$ and $K = 0.01$.

5. Results and discussion

Lie-group method is applicable to both linear and nonlinear partial differential equations, which yields similarity variables that may be used to reduce the number of independent variables in partial differential equations. By determining the transformation group under which the given partial differential equation is invariant, we can obtain information about the invariants and symmetries of that equation. This information can be used to determine similarity variables that will reduce the number of independent variables in the system. In this paper, we have used Lie-symmetry techniques to obtain similarity reductions of a nonlinear gas diffusion equation in nuclear fission (2.1) with the assumption that $D(C) = D_0C^n$. By determining the transformation group under which the given partial differential equation is invariant, we obtained information about the invariants and symmetries of that equation. This information, in turn, is used to determine similarity variables that reduced the number of independent variables by one. The ordinary differential equation (3.18) with the appropriate corresponding conditions (3.19) is solved numerically using nonlinear finite difference method for different values of the index n, the dimensionless group constant K , the time τ, and the grain radius. By comparing our results with those obtained by Abd-el-Malek *et al*. [6] for $n = -1$, they were found in complete agreement.

Lie-group method is more general than any other group methods that start out with an assumed form of a group that limits the generality of the results. We concluded that after doing a comparison between Lie-group method applied in this work and the transformation group theoretic method applied by Abd-el-Malek *et al*. [6]. As seen, the solution obtained from transformation group theoretic method is one of the solutions obtained by applying Lie-group method to the same partial differential equation.

References

1. Hayns MR, Wood MH. Models of fission gas behavior in fast reactor fuels under steady state and transient conditions. *Journal of Nuclear Materials* 1977; **67**:155–170.
2. Mac Innes DA, Brearley IR. A model for the release of fission gas from reactor fuel undergoing transient heating. *Journal of Nuclear Materials* 1982; **107**:123–132.
3. Matthews JR, Wood MH. Solution of the problem of diffusion flow to a spherical boundary. *Nuclear Engineering and Design* 1980; **56**:439–443.
4. Dowling DM, White RJ, Tucker MO. The effect of irradiation-induced re-solution on fission gas release. *Journal of Nuclear Materials* 1982; **110**:37–46.
5. Rest J. An improved model for fission product behavior in nuclear fuel under normal and accident conditions. *Journal of Nuclear Materials* 1984; **120**:195–212.
6. Abd-el-Malek MB, Badran NA, Hassan HS. Using group theoretic method for fission product behavior in nuclear fuel. *International Journal of Applied Mathematics* 2001; **7**:333–348.
7. Sophocleous C. On symmetries of radially nonlinear symmetric diffusion equations. *Journal of Mathematical Physics* 1992; **33**:3687–3693.
8. Vaneeva OO, Johnpillai AG, Popovych RO, Sophocleos C. Enhanced group analysis and conservation laws of variable coefficient reaction-diffusion equations with power nonlinearities. *Journal of Mathematical Analysis and Applications* 2007; **330**(2):1363–1386.
9. Vaneeva OO, Popovych RO, Sophocleos C. Enhanced group analysis and exact solutions of variable coefficient semilinear diffusion equations with a power source. *Acta Applicandae Mathematicae* 2009; **106**(1):1–46.
10. Hill JM. *Solution of Differential Equations by Means of One-Parameter Groups*. Pitman Publishing Inc.: Massachusetts, USA, 1982.
11. Seshadri R, Na TY. *Group Invariance in Engineering Boundary Value Problems*. Springer-Verlag: New York, 1985.
12. Ibragimov NH. *Elementary Lie Group Analysis and Ordinary Differential Equations*. Wiley: New York, 1999.
13. Hydon PE. *Symmetry Methods for Differential Equations*. CUP: Cambridge, 2000.
14. Burde GI. Expanded Lie group transformations and similarity reductions of differential equations. *Proceedings of Institute of Mathematics of NAS of Ukraine* 2002; **43**:93–101.
15. Olver PJ. *Applications of Lie Groups to Differential Equations*. Springer-Verlag: New-York, 1986.
16. Boutros YZ, Abd-el-Malek MB, Badran NA, Hassan HS. Lie-group method for unsteady flows in a semi-infinite expanding or contracting pipe with injection or suction through a porous wall. *Journal of Computational and Applied Mathematics* 2006; **197**:465–494.
17. Boutros YZ, Abd-el-Malek MB, Badran NA, Hassan HS. Lie-group method of solution for steady two-dimensional boundary-layer stagnation-point flow towards a heated stretching sheet placed in a porous medium. *Meccanica* 2007; **41**(6):681–691.
18. Boutros YZ, Abd-el-Malek MB, Badran NA, Hassan HS. Lie-group method solution for two-dimensional viscous flow between slowly expanding or contracting walls with weak permeability. *Journal of Applied Mathematical Modelling* 2007; **31**:1092–1108.
19. Abd-el-Malek MB, Badran NA, Hassan HS. Lie-group method for predicting water content for immiscible flow of two fluids in a porous medium. *Applied Mathematical Sciences* 2007; **1**(24):1169–1180.
20. Abd-el-Malek MB, Hassan HS. Internal flow through a conducting thin duct via symmetry analysis. *Proceedings of the International Conference on SPT 2007*, Otranto, Italy, 2-9 June 2007; 233–234.
21. Abd-el-Malek MB, Hassan HS. Symmetry analysis for steady boundary-layer stagnation-point flow of Rivlin–Ericksen fluid of second grade subject to suction. *Nonlinear Analysis: Modelling and Control* 2010; **15**(4):379–396.
22. Dowling DM. The effect of irradiation-induced re-solution on fission gas release. *Journal of Nuclear Materials* 1982; **110**:37–46.
23. Pucci E, Saccomandi G. Potential symmetries and solutions by reduction of partial differential equations. *Journal of Physics A: Mathematical and General* 1993; **26**:681–690.