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A Study for MHD Boundary Layer Flow of Variable Viscosity over a Heated Stretching Sheet via Lie-Group Method

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Abstract: The present work deals with the study of Magnetohydrodynamic (MHD) boundary layer flow over a heated stretching sheet with variable fluid viscosity. The fluid viscosity is assumed to vary as a linear function of temperature in the presence of uniform transverse magnetic field. The fluid is assumed to be electrically conducting. Lie-group method is applied for determining symmetry reductions for the MHD boundary-layer equations. Lie-group method starts out with a general infinitesimal group of transformations under which the given partial differential equations are invariant. The determining equations are a set of linear differential equations, the solution of which gives the transformation function or the infinitesimals of the dependent and independent variables. After the group has been determined, a solution to the given partial differential equations may be found from the invariant surface condition such that its solution leads to similarity variables that reduce the number of independent variables of the system. The effect of the Hartmann number (M), the viscosity parameter (A) and the Prandtl number (Pr) on the horizontal and vertical velocities, temperature profiles, wall heat transfer and the wall shear stress (skin friction), have been studied and the results are plotted.

Keywords: Lie-group; Similarity solutions; MHD flow; Stretching sheet; Tempreture-dependent fluid viscosity
MSC: 76W05; 76M60; 35Q30

1 Introduction

Flow and heat transfer of an incompressible viscous fluid over a stretching sheet appear in several industrial processes such as the extrusion of polymers, the cooling of metallic plates, the aerodynamic extrusion of plastic sheets, etc. In the glass industry, blowing, floating or spinning of fibers are processes, which involve the flow due to a stretching surface, [1]. The study of heat transfer and flow field is necessary for determining the quality of the final products of such processes.

Sakiadis [2] presented the pioneering work in this field. He investigated the flow induced by a semi-infinite horizontally moving wall in an ambient fluid.

Crane [3] studied the flow over a linearly stretching sheet in an ambient fluid and gave a similarity solution in closed analytical form for the steady two-dimensional problem. He presented a closed form exponential solution for the planar viscous flow of linear stretching case.

Moreover, the study of Magnetohydrodynamic (MHD) flow of an electrically conducting fluid is of considerable interest in modern metallurgical and metal-working processes. There has been a great interest in the study of magnetohydrodynamic flow and heat transfer in any medium due to the effect of magnetic field on the boundary layer flow control and on the performance of many systems using electrically conducting fluids, [4].

The problem of flow, heat and mass transfer over a stretching sheet in the presence of suction or blowing was examined by Gupta and Gupta [5]. Dutta and Gupta [6] extended the pioneering works of Crane [3] to explore various aspects of the flow and heat transfer occurring in

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an infinite domain of the fluid surrounding the stretching sheet.

The fluid viscosity was assumed uniform in the flow region in all the pervious mentioned works. Practically, the coefficient of viscosity decreases in case of liquids whereas it increases in case of gases as the temperature increases.

Pop et al. [7] and Pantokratoras [8] studied the effect of variable viscosity on flow and heat transfer to a continuous moving flat plate. In their work, they assumed that the fluid viscosity varies as an inverse function of temperature.

Abel et al. [9] studied the boundary layer flow and heat transfer of a visco-elastic fluid immersed in a porous over a stretching sheet with variable fluid viscosity. The fluid viscosity is assumed to vary as an inverse function of temperature. A numerical shooting algorithm with fourth-order Runge-Kutta integration scheme has been used to solve the coupled nonlinear boundary value problem.

Mukhopadhyay et al. [4] studied the MHD boundary layer flow over a heated stretching sheet with variable viscosity. The fluid viscosity is assumed to vary as a linear function of temperature. The scaling group of transformations is applied to the governing equations. The resulting system of non-linear ordinary differential equations is solved numerically.

Pantokratoras [10] critiqued the work of Mukhopadhyay et al. [4]. He concluded that, in the work of Mukhopadhyay et al. [4], the calculation domain was small and the temperature profiles are truncated. The results of his work are obtained with the direct numerical solution of the boundary layer equations taking into account both viscosity and Prandtl number variation across the boundary layer. The temperature profiles of his work are quite different from those of Mukhopadhyay et al. [4].

This paper is concerned with the study of MHD boundary layer flow over a heated stretching sheet with variable fluid viscosity using Lie-group method. Following Batchelor [11], the fluid viscosity is assumed to vary as a linear function of temperature in the presence of uniform transverse magnetic field. The fluid is assumed to be electrically conducting. Lie-group theory is applied to the equations of motion and energy for determining symmetry reductions of partial differential equations, [12–25]. The resulting system of non-linear differential equations is then solved numerically using shooting method coupled with Runge-Kutta scheme. The obtained results are compared with those of Mukhopadhyay et al. [4], Pantokratoras [10], Chiam [26], Carragher and Crane [27], Grubka and Bobba [28].

2 Mathematical Formulation of the Problem

Consider a steady, two-dimensional flow of a viscous and incompressible electrically conducting fluid over a heated stretching sheet placed in the region $\bar{y} > 0$ of a Cartesian

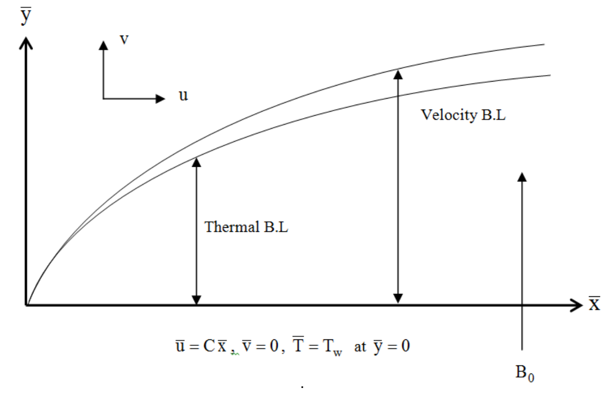

Fig. 1: Physical model and coordinate system.

system of coordinates $O\bar{x}\bar{y}$. The stretching surface has a uniform temperature T_w and the free stream temperature is T_∞ with $T_w > T_\infty$. The wall is stretched by applying two equal and opposite forces along the $\bar{x}-$ axis, to keep the origin fixed. A uniform magnetic field of strength B_0 is assumed to be imposed along the $\bar{y}-$ axis. The viscosity of the fluid is assumed to be temperature-dependent.

Under the above assumptions, the resulting boundary-layer equations are given by:

Continuity Equation:

$$\frac{\partial \bar{u}}{\partial \bar{x}} + \frac{\partial \bar{v}}{\partial \bar{y}} = 0, \tag{2.1}$$

Momentum Equation:

$$\bar{u}\frac{\partial \bar{u}}{\partial \bar{x}} + \bar{v}\frac{\partial \bar{u}}{\partial \bar{y}} = \frac{1}{\rho}\frac{\partial \bar{\mu}}{\partial \bar{T}}\frac{\partial \bar{T}}{\partial \bar{y}}\frac{\partial \bar{u}}{\partial \bar{y}} + \frac{\bar{\mu}}{\rho}\frac{\partial^2 \bar{u}}{\partial \bar{y}^2} - \frac{\sigma {B_0}^2}{\rho}\bar{u}, \tag{2.2}$$

Energy Equation:

$$\bar{u}\frac{\partial \bar{T}}{\partial \bar{x}} + \bar{v}\frac{\partial \bar{T}}{\partial \bar{y}} = \alpha \frac{\partial^2 \bar{T}}{\partial \bar{y}^2}, \tag{2.3}$$

and with the following boundary conditions,

$$\begin{aligned} &(i) \quad \bar{u} = C\bar{x}, \quad \bar{v} = 0, \quad \bar{T} = T_w \quad \text{at} \quad \bar{y} = 0, \\ &(ii) \quad \bar{u} \to 0, \qquad\qquad \bar{T} \to T_\infty \quad \text{as} \quad \bar{y} \to \infty, \end{aligned} \tag{2.4}$$

where $\bar{u}$ and $\bar{v}$, are the velocity components in the $\bar{x}$ and $\bar{y}$ directions, respectively, $\bar{T}$ is the temperature and $\bar{\mu}$ is the temperature dependent viscosity of the fluid. Moreover, ρ is the fluid density, σ is the electrical conductivity of the fluid, B_0 is the strength of uniform magnetic field, α is the coefficient of the thermal diffusivity, C is a constant, is a constant, T_w is the wall temperature and T_∞ is the free stream temperature.

Follow Batchelor [11], the temperature dependent viscosity is assume to be in the form

$$\bar{\mu} = \mu^* \left(a + b\left(T_w - \bar{T}\right)\right), \tag{2.5}$$

where μ^* is the reference viscosity, and are constants.

The variables in equations (2.1) - (2.4) are dimensionless according to

$$x=\frac{C\bar{x}}{U_1},\ y=\sqrt{\frac{C}{\nu}}\bar{y},\ u=\frac{\bar{u}}{U_1},\ v=\frac{\bar{v}}{\sqrt{C\nu}},\ T=\frac{\bar{T}-T_\infty}{T_w-T_\infty},\tag{2.6}$$

where U_1 is the characteristic velocity and $\nu=\mu^*/\rho$ is the kinematic viscosity.

Substitution from (2.5)-(2.6) into (2.1) - (2.3), gives

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0,\tag{2.7}$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-A\frac{\partial T}{\partial y}\frac{\partial u}{\partial y}+(a+A(1-T))\frac{\partial^2 u}{\partial y^2}-M^2u,\tag{2.8}$$

$$u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\frac{1}{\Pr}\frac{\partial^2 T}{\partial y^2},\tag{2.9}$$

where, $M^2=\frac{\sigma B_0^2}{\rho C}$, M is the Hartmann number (constant), $A=b(T_w-T_\infty)$ is the viscosity parameter (constant) and $\Pr=\frac{\nu}{\alpha}$ is the Prandtl number.

The boundary conditions (2.4) will be considered as follows,

$$\begin{aligned}&(i)\quad u=x,\quad v=0,\quad T=1\quad \text{at}\ y=0,\\ &(ii)\quad u\to 0,\qquad T\to 0\quad \text{as}\quad y\to\infty.\end{aligned}\tag{2.10}$$

From the continuity equation (2.7), a stream function $\Psi(x,y)$ may exist as,

$$u(x,y)=\frac{\partial\Psi}{\partial y},\ v(x,y)=-\frac{\partial\Psi}{\partial x},\tag{2.11}$$

which satisfies equation (2.7) identically.

Substituting from (2.11) into (2.8)-(2.9) yields

$$\Psi_y\Psi_{xy}-\Psi_x\Psi_{yy}+A\,T_y\Psi_{yy}-(a+A(1-T))\,\Psi_{yyy}+M^2\,\Psi_y=0,\tag{2.12}$$

and

$$\Psi_yT_x-\Psi_xT_y-\frac{1}{\Pr}T_{yy}=0,\tag{2.13}$$

where subscripts denote partial derivatives.

The boundary conditions (2.10) will be as follows,

$$\begin{aligned}&(i)\quad \Psi_y=x,\quad \Psi_x=0,\quad T=1\quad \text{at}\ y=0,\\ &(ii)\quad \Psi_y\to 0,\qquad T\to 0\quad \text{as}\quad y\to\infty.\end{aligned}\tag{2.14}$$

3 Solution of the Problem

Firstly, we derive the similarity solutions using Lie-group method under which (2.12)-(2.13) and the boundary conditions (2.14) are invariant, and then we use these symmetries to determine the similarity variables.

3.1 Lie Point Symmetries

Consider the one-parameter (ε) Lie group of infinitesimal transformations in $(x,y;\Psi,T)$ given by

$$\begin{aligned}x^*&=x+\varepsilon\,\varphi(x,y;\Psi,T)+O(\varepsilon^2),\\ y^*&=y+\varepsilon\,\zeta(x,y;\Psi,T)+O(\varepsilon^2),\\ \Psi^*&=\Psi+\varepsilon\,\eta(x,y;\Psi,T)+O(\varepsilon^2),\\ T^*&=T+\varepsilon\,F(x,y;\Psi,T)+O(\varepsilon^2),\end{aligned}\tag{3.1}$$

where "ε" is a small parameter.

A system of partial differential equations (2.12)-(2.13) is said to admit a symmetry generated by the vector field

$$X\equiv\varphi\frac{\partial}{\partial x}+\zeta\frac{\partial}{\partial y}+\eta\frac{\partial}{\partial\Psi}+F\frac{\partial}{\partial T},\tag{3.2}$$

if it is left invariant by the transformation $(x,y;\Psi,T)\to(x^*,y^*;\Psi^*,T^*)$.

The solutions $\Psi=\Psi(x,y)$ and $T=T(x,y)$, are invariant under the symmetry (3.2) if

$$\Phi_\Psi=X(\Psi-\Psi(x,y))=0\ \text{when}\ \Psi=\Psi(x,y),\tag{3.3}$$

and

$$\Phi_T=X(T-T(x,y))=0\text{when}T=T(x,y).\tag{3.4}$$

Assume,

$$\Delta_1=\Psi_y\Psi_{xy}-\Psi_x\Psi_{yy}+A\,T_y\Psi_{yy}-(a+A(1-T))\,\Psi_{yyy}+M^2\,\Psi_y,\tag{3.5}$$

$$\Delta_2=\Psi_yT_x-\Psi_xT_y-\frac{1}{\Pr}T_{yy}.\tag{3.6}$$

A vector X given by (3.2), is said to be a Lie point symmetry vector field for (2.12)-(2.13) if

$$X^{[3]}(\Delta_j)\big|_{\Delta_j=0}=0,\ j=1,2,\tag{3.7}$$

where,

$$\begin{aligned}X^{[3]}\equiv{}&\varphi\tfrac{\partial}{\partial x}+\zeta\tfrac{\partial}{\partial y}+\eta\tfrac{\partial}{\partial\Psi}+F\tfrac{\partial}{\partial T}+\eta^x\tfrac{\partial}{\partial\Psi_x}+\eta^y\tfrac{\partial}{\partial\Psi_y}\\ &+F^x\tfrac{\partial}{\partial T_x}+F^y\tfrac{\partial}{\partial T_y}+\eta^{xy}\tfrac{\partial}{\partial\Psi_{xy}}+\eta^{yy}\tfrac{\partial}{\partial\Psi_{yy}}+F^{yy}\tfrac{\partial}{\partial T_{yy}}+\eta^{yyy}\tfrac{\partial}{\partial\Psi_{yyy}},\end{aligned}\tag{3.8}$$

is the third prolongation of X.

To calculate the prolongation of the given transformation, we need to differentiate (3.1) with respect to each of the variables, and . To do this, we introduce the following total derivatives

$$\begin{aligned}D_x&\equiv\partial_x+\Psi_x\partial_\Psi+T_x\partial_T+\Psi_{xx}\partial_{\Psi_x}+T_{xx}\partial_{T_x}+\Psi_{xy}\partial_{\Psi_y}+\ldots\ldots,\\ D_y&\equiv\partial_y+\Psi_y\partial_\Psi+T_y\partial_T+\Psi_{yy}\partial_{\Psi_y}+T_{yy}\partial_{T_y}+\Psi_{xy}\partial_{\Psi_x}+\ldots\ldots\end{aligned}\tag{3.9}$$

Table 1: Table of commutators of the basis operators.

	X_1	X_2	X_3
X_1	0	0	$-X_3$
X_2	0	0	0
X_3	X_3	0	0

Table 2: Solution of the invariant surface conditions.

Generator	Characteristic $\Phi = (\Phi_\Psi, \Phi_T)$	Solution of the invariant surface conditions
X_1	$\Phi_\Psi = \Psi - x\,\Psi_x$, $\Phi_T = -T_x$	$\Psi = x\,G(y), T = T(y)$
X_2	$\Phi_\Psi = -\Psi_y$, $\Phi_T = -T_y$	$\Psi = \Psi(x)$, $T = T(x)$
X_3	$\Phi_\Psi = 1$, $\Phi_T = 0$	No solution
$X_1 + \beta X_2$	$\Phi_\Psi = \Psi - x\,\Psi_x - \beta\Psi_y$, $\Phi_T = -xT_x - \beta\,T_y$	$\Psi = x\,L_1(\omega), T = T(\omega)$, $\omega = y - \ln x^\beta$
$X_1 + \beta X_3$	$\Phi_\Psi = \Psi + \beta - x\,\Psi_x$, $\Phi_T = -xT_x$	$\Psi = x\,L_2(y) - \beta$, $T = T(y)$
$X_2 + \beta X_3$	$\Phi_\Psi = \beta - \Psi_y$, $\Phi_T = -T_y$	$\Psi = \beta y + h(x), T = T(x)$
$X_1 + \lambda X_2 + \delta X_3$	$\Phi_\Psi = \Psi + \delta - x\,\Psi_x - \lambda\,\Psi_y$, $\Phi_T = -xT_x - \lambda\,T_y$	$\Psi + \delta = x\,L_2(\pi), T = T(\pi)$, $\pi = y - \ln x^\lambda$

Equation (3.7) gives the following system of linear partial differential equations

$$A\,F\,\Psi_{yyy} - \Psi_{yy}\,\eta^x + (\Psi_{xy} + M^2)\,\eta^y + A\,\Psi_{yy}\,F^y + \Psi_y\eta^{xy} + (A\,T_y - \Psi_x)\eta^{yy} - (a + A(1-T))\;\eta^{yyy} = 0, \tag{3.10}$$

and

$$-T_y\,\eta^x + T_x\,\eta^y + \Psi_y F^x - \Psi_x F^y - \frac{1}{\Pr} F^{yy} = 0. \tag{3.11}$$

The components $\eta^x, \eta^y, F^x, F^y, \eta^{xy}, \eta^{yy}, F^{yy}, \eta^{yyy}$ can be determined from the following expressions

$$\eta^S = D_S\eta - \Psi_x D_S\varphi - \Psi_y D_S\zeta\,, F^S = D_S F - T_x D_S\varphi - T_y D_S\zeta,$$
$$\eta^{JS} = D_S\eta^J - \Psi_{Jx} D_S\varphi - \Psi_{Jy} D_S\zeta, F^{JS} = D_S F^J - T_{Jx} D_S\varphi - T_{Jy} D_S\zeta, \tag{3.12}$$

where S and J are standing for x, y.

Substitution from (3.12) into (3.10)-(3.11) and solving the resulting determining equations in view of the invariance of the boundary conditions (2.14), yields

$$\varphi = C_1\,x, \qquad \zeta = C_2, \qquad \eta = C_1\Psi + C_3, \qquad F = 0. \tag{3.13}$$

So, the nonlinear equations (2.12)-(2.13) have the three-parameter Lie group of point symmetries generated by

$$X_1 \equiv x\frac{\partial}{\partial x} + \Psi\frac{\partial}{\partial \Psi}, \quad X_2 \equiv \frac{\partial}{\partial y}, \quad \text{and} \quad X_3 \equiv \frac{\partial}{\partial \Psi}. \tag{3.14}$$

The one-parameter group generated by X_1 consists of scaling, whereas X_2 and X_3 consists of translation. The commutator table of the symmetries is given in Table1, where the entry in the ith row and jth column is defined as $X_i, X_j] = X_iX_j - X_jX_i$.

The finite transformations corresponding to the symmetries X_1, X_2 and X_3 are respectively

$$\begin{aligned} X_1: &\quad x^* = e^{\varepsilon\,1}x, \quad y^* = y, \quad \Psi^* = e^{\varepsilon\,1}\Psi, \quad T^* = T, \\ X_2: &\quad x^* = x, \quad y^* = y + \varepsilon_2, \quad \Psi^* = \Psi, \quad T^* = T, \\ X_3: &\quad x^* = x, \quad y^* = y, \quad \Psi^* = \Psi + \varepsilon_3, \quad T^* = T, \end{aligned} \tag{3.15}$$

where ε_1, ε_2 and ε_3 are group parameters.

Table 2 shows the solution of the invariant surface conditions (3.3)-(3.4).

For X_1 , the characteristic,

$$\Phi = (\Phi_\Psi, \Phi_T)\,, \tag{3.16}$$

has the component

$$\Phi_\Psi = \Psi - x\,\Psi_x, \qquad \Phi_T = -T_x. \tag{3.17}$$

Therefore, the general solutions of the invariant surface conditions (3.3)-(3.4) are

$$\Psi = x\,G(y)\,, \qquad T = T(y). \tag{3.18}$$

Substitution from (3.18) into (2.12)-(2.13) yields

$$\left(\frac{dG}{dy}\right)^2 - G\frac{d^2G}{dy^2} + A\,\frac{dT}{dy}\,\frac{d^2G}{dy^2} - (a + A(1-T))\,\frac{d^3G}{dy^3} + M^2\,\frac{dG}{dy} = 0, \tag{3.19}$$

and

$$\frac{d^2T}{dy^2} + \Pr\,G\,\frac{dT}{dy} = 0. \tag{3.20}$$

The boundary conditions (2.14) will be

$$\begin{aligned} (i)\ & \frac{dG}{dy} = 1, \quad G = 0, \quad T = 1 \quad \text{as}\ \ y = 0, \\ (ii)\ & \frac{dG}{dy} \to 0 \qquad\qquad T \to 0 \quad \text{as}\ \ y \to \infty. \end{aligned} \tag{3.21}$$

For X_2, the characteristic (3.16) has the component

$$\Phi_\Psi = -\Psi_y, \ \Phi_T = -T_y. \tag{3.22}$$

Therefore, the general solutions of the invariant surface conditions (3.3)-(3.4) are $\Psi = \Psi(x)$ and $T = T(x)$, which contradict the boundary conditions.

For X_3, the characteristic (3.16) has the component

$$\Phi_\Psi = 1, \ \Phi_T = 0. \tag{3.23}$$

Therefore, no solution invariant under X_3.

For $X_1 + \beta X_2$, the characteristic (3.16) has the component

$$\Phi_\Psi = \Psi - x\,\Psi_x - \beta\,\Psi_y, \quad \Phi_T = -xT_x - \beta\,T_y. \tag{3.24}$$

Therefore, the general solutions of the invariant surface conditions (3.3)-(3.4) are $\Psi = x\,L_1(\omega)$ and $T = T(\omega)$, where $\omega = y - \ln x^\beta$ is the similarity variable. These solutions contradict the boundary conditions.

Table 3: Values of the wall heat transfer $(-T'(0))$ for different values of M and Pr at $A = 0.0$

M	$-T'(0)$				
	Pr				
	0.40	0.60	1.00	10.0	20.0
0.0	0.2992074	0.4059944	0.5819919	2.3080132	3.3539139
1.0	0.2546717	0.3409365	0.5017490	2.2175688	3.2640175

For $X_1 + \beta X_3$, the characteristic (3.16) has the component

$$\Phi_\Psi = \Psi + \beta - x\,\Psi_x, \qquad \Phi_T = -xT_x. \tag{3.25}$$

Therefore, the general solutions of the invariant surface conditions (3.3)-(3.4) are

$$\Psi = xL_2(y) - \beta, \qquad T = T(y). \tag{3.26}$$

Substitution from (3.26) into (2.12)-(2.13) yields the same ordinary differential equations (3.19)-(3.20) with the same boundary conditions (3.21). So, the solutions invariant under both X_1 and $X_1 + \beta X_3$ are the same.

For $X_2 + \beta X_3$, , the characteristic (3.16) has the component

$$\Phi_\Psi = \beta - \Psi_y, \qquad \Phi_T = -T_y. \tag{3.27}$$

Therefore, the general solutions of the invariant surface conditions (3.3)-(3.4) are $\Psi = \beta y + h(x)$ and $T = T(x)$, which contradicts the boundary conditions.

For $X_1 + \lambda X_2 + \delta X_3$, , the characteristic (3.16) has the component

$$\Phi_\Psi = \Psi + \delta - x\,\Psi_x - \lambda\,\Psi_y, \qquad \Phi_T = -xT_x - \lambda\,T_y. \tag{3.28}$$

Therefore, the general solutions of the invariant surface conditions (3.3)-(3.4) are $\Psi + \delta = x\,L_2(\pi)$ and $T = T(\pi)$, where $\pi = y - \ln x^\lambda$ is the similarity variable. These solutions contradict the boundary conditions.

3.2 Numerical Solution

The system of non-linear differential equations (3.19)-(3.20) with the boundary conditions (3.21) is solved numerically using the shooting method, coupled with Runge-Kutta scheme. We take $a = 1$, in all calculations.

From (2.11) and (3.18), we get

$$\frac{u}{x} = \frac{dG}{dy}, \qquad v = -\,G(y), \qquad T = T(y). \tag{3.29}$$

4 Results and Discussion

4.1 Horizontal Velocity

4.1.1 The Effect of the Hartmann Number M

Figure 2 illustrates the behaviour of the horizontal velocity u/x for $\Pr = 0.1$ and $\Pr = 1.0$ with $A = 0$, over a range of the Hartmann number M. As seen, the horizontal velocity increases by decreasing M, i.e. the transport rate increases, which indicate that the transverse magnetic field is opposite to the transport phenomena. That is because, variation of the Hartmann number leads to the variation of the Lorentz force due to the transverse magnetic field and accordingly, the Lorentz force produces more resistance to transport phenomena, [4]. Our profiles are agreed very well with those of Mukhopadhyay et al. [4] and Pantokratoras [10]. As seen from Figure 2b, the calculation domain is higher than that of Figure 2a, that is because the value of the Prandtl number is increases.

4.1.2 The Effect of the Viscosity Parameter A

Figure 3 illustrates the behaviour of the horizontal velocity u/x for $\Pr = 0.1$ and $\Pr = 1.0$ with $M = 0$, over a range of the viscosity parameter A. As seen, the horizontal velocity increases by increasing A. That is because, with increasing A, the fluid viscosity decreases resulting in increment of the velocity boundary layer thickness. Unfortunately, in the work of Mukhopadhyay et al. [4] and Pantokratoras [10], the profiles are reversed, i.e. the horizontal velocity decreases as A increases, which is wrong.

4.2 Vertical Velocity

4.2.1 The Effect of the Hartmann Number M

Figure 4 illustrates the behaviour of the vertical velocity v for $\Pr = 0.1$ and $\Pr = 1.0$ with $A = 0$, over a range of the Hartmann number M. As seen, the absolute value of the vertical velocity increases by decreasing M as mentioned before. These profiles were not present in both works of Mukhopadhyay et al. [4] and Pantokratoras [10].

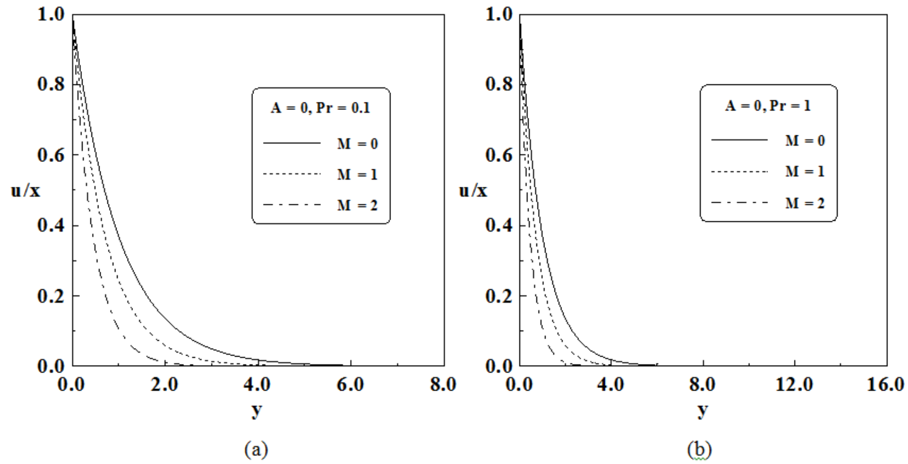

Fig. 2: Horizontal velocity profiles over a range of M with $A=0$ at: (a) $\mathrm{Pr}=0.1$ (b) $\mathrm{Pr}=1.0$.

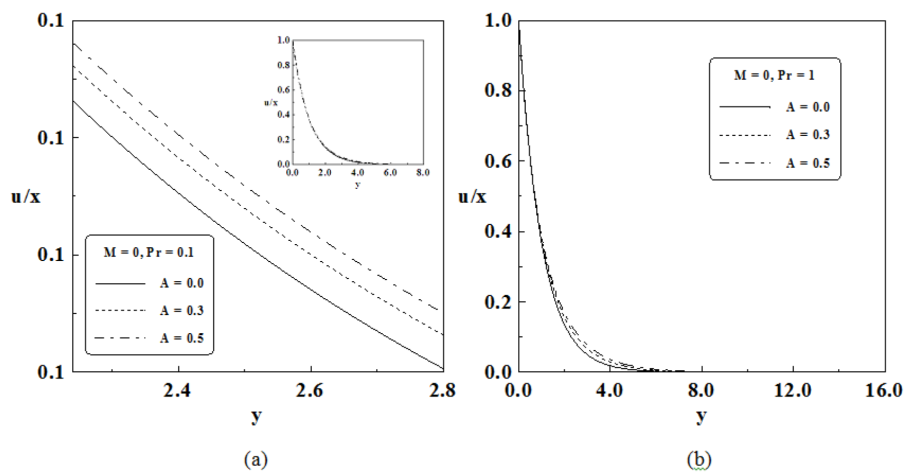

Fig. 3: Horizontal velocity profiles over a range of A with $M=0$ at: (a) $\mathrm{Pr}=0.1$ (b) $\mathrm{Pr}=1.0$.

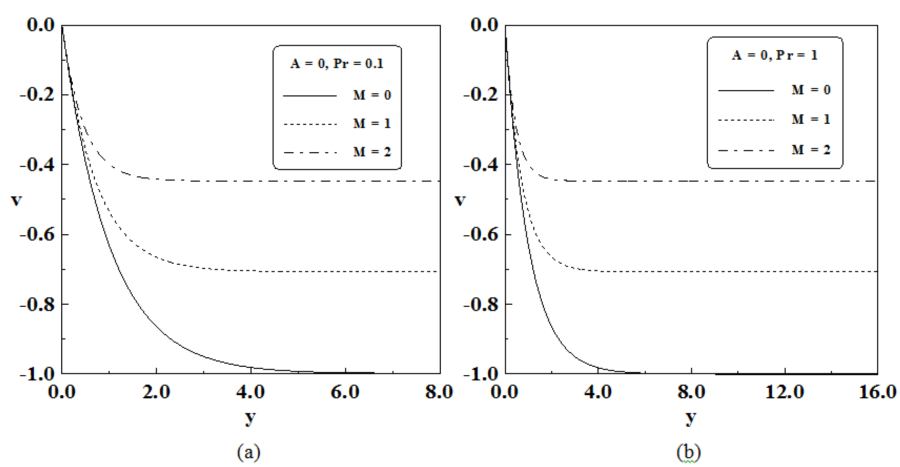

Fig. 4: Vertical velocity profiles over a range of M with $A=0$ at: (a) $\mathrm{Pr}=0.1$ (b) $\mathrm{Pr}=1.0$.

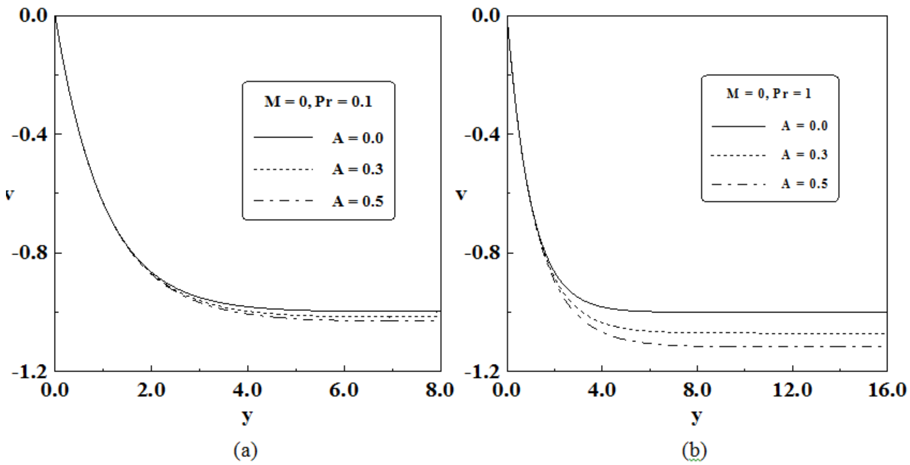

Fig. 5: Vertical velocity profiles over a range of A with $M = 0$ at: (a) $\mathrm{Pr} = 0.1$ (b) $\mathrm{Pr} = 1.0$.

4.2.2 The Effect of the Viscosity Parameter A

Figure 5 illustrates the behaviour of the vertical velocity v for $\mathrm{Pr} = 0.1$ and $\mathrm{Pr} = 1.0$ with $M = 0$, over a range of the viscosity parameter A, As seen, the absolute value of the vertical velocity increases by increasing A. Also, these profiles were not present in both works of Mukhopadhyay et al. [4] and Pantokratoras [10].

4.3 The Temperature Profiles

4.3.1 The Effect of the Hartmann Number M

Figure 6 illustrates the variation of the temperature profiles T for $\mathrm{Pr} = 0.1$ with $A = 0$, over a range of the Hartmann number M. We notice that, the temperature increases as M increases and therefore the thinning of the thermal boundary layer. Our results are in complete agreement with that reported by Mukhopadhyay et al. [4], but in their work, the calculation domain is small which causes the temperature profiles appear truncated. This disadvantage was critiqued by Pantokratoras [10]. Unfortunately, in the work of Pantokratoras [10], the profiles are reversed, i.e. the temperature increases as M decreases, which is wrong.

4.3.2 The Effect of the Viscosity Parameter A

Figure 7 illustrates the variation of the temperature profiles T for $\mathrm{Pr} = 1.0$ with $M = 0$, over a range of the viscosity parameter A. As seen, the temperature decreases as A increases. That is because, the increase of the viscosity parameter A causes decrease of the thermal boundary layer thickness which results in decrease of the temperature. Our results are in complete agreement with that reported by Mukhopadhyay et al. [4], but unfortunately in their work, the profiles are reversed; see Figure 2 in their work. The same mistake appeared in the work of Pantokratoras [10]; see Figure 3 in his work.

4.3.3 The Effect of the Prandtl Number Pr

Figure 8 illustrates the variation of the temperature profiles T for $M = 0$ and $M = 1$ with $A = 0$, over a range of Prandtl number Pr. It is noticed that, as Pr decreases, the thickness of the thermal boundary layer becomes greater than the thickness of the velocity boundary layer according to the well known relation $\frac{\delta_T}{\delta} \approx (\mathrm{Pr})^{-1/2}$, where δ_T is the thickness of the thermal boundary layer and δ is the thickness of the velocity boundary layer. So, the thickness of the thermal boundary layer increases as Pr decreases and hence, the temperature T increases with the decrease of Pr

4.4 Wall Heat Transfer

When the Prandtl number increases, the thickness of thermal boundary layer becomes thinner and this causes an increase in the gradient of the temperature. Therefore, the wall heat transfer $(-T'(0))$ increases as Pr increases. For different values of the Hartmann number M and Prandtl number Pr at $A = 0.0$, values of the wall heat transfer are computed, Table 3. Also, for fixed value of Pr , the wall heat transfer $(-T'(0))$ decreases as the Hartmann number M increases as mentioned before. The value of $(-T'(0))$ is positive which is consistent with the fact that the heat flows from the surface to the fluid as long as $T_w > T_\infty$ in the absence of viscous dissipation.

The computed values of $(-T'(0))$ are compared with those obtained by Chiam [26], Carragher and Crane [27],

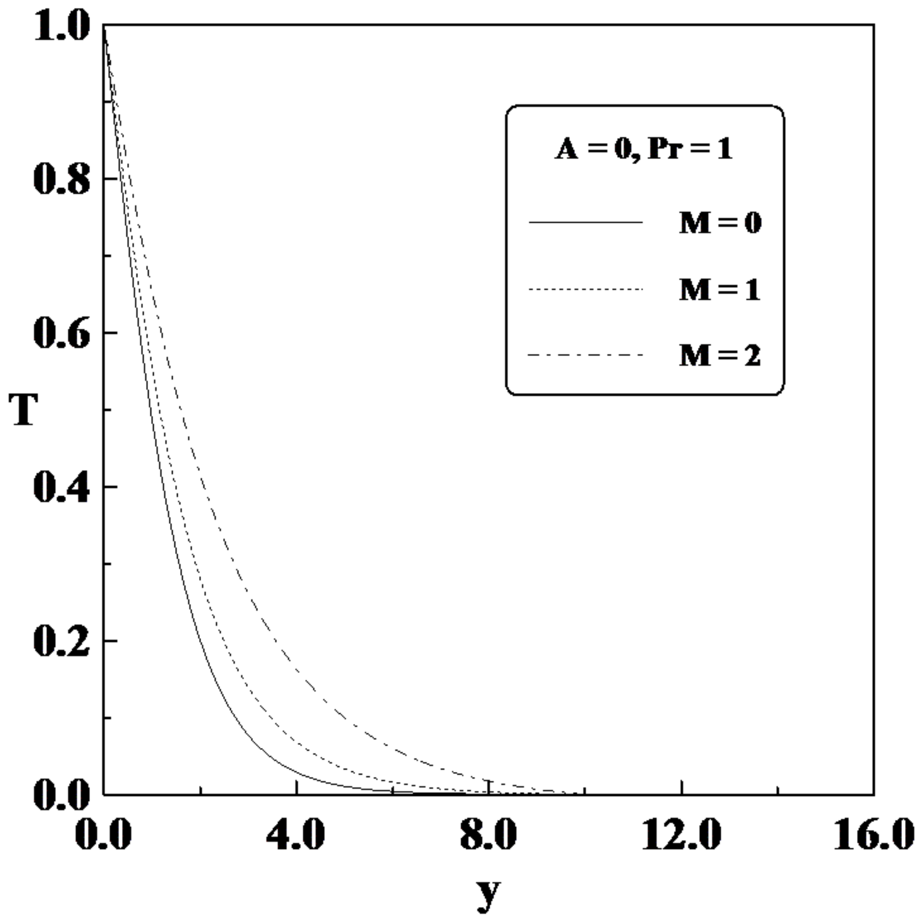

Fig. 6: Temperature profiles over a range of M with $\mathrm{Pr} = 1.0$ and $A = 0$

Table 4: Comparison between the values of $(-T'(0))$ at $A = 0.0$ and $M = 0$ for different values of Pr

Pr	$-T'(0)$				
	Chiam [26]	Carragher and Crane [27]	Grubka and Bobba [28]	Pantokratoras [10]	Present work
0.023	0.022489			0.0240	0.0225438
0.100	0.091292			0.0925	0.0924952
0.700		0.46		0.4543	0.4539453
1.000	0.581977			0.5820	0.5819919
10.00			2.3080	2.3080	2.3080132

Table 5: Values of the dimensionless wall shear stress $G''(0)$ for different M and A at $\mathrm{Pr} = 1.0$

M	$G''(0)$		
	A		
	0.0	0.3	0.5
0.0	-0.9999893	-1.0290572	-1.0483274
1.0	-1.4142136	-1.4455980	-1.4670111

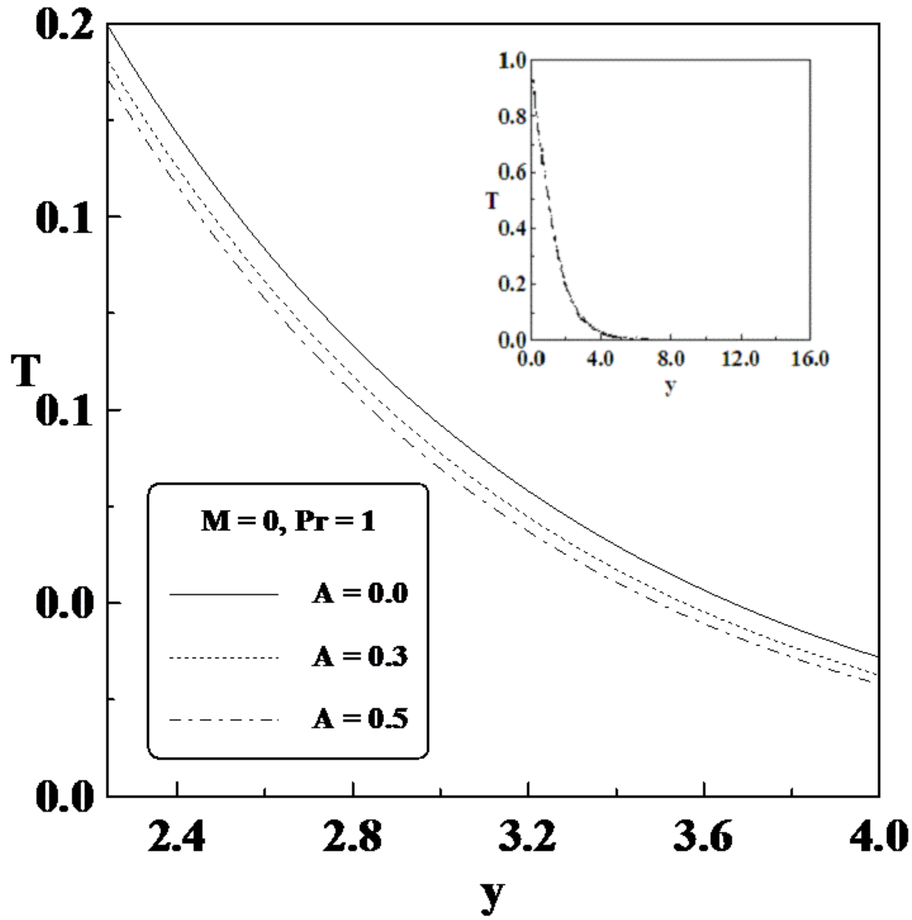

Fig. 7: Temperature profiles over a range of A with $\mathrm{Pr} = 1.0$ and $M = 0$

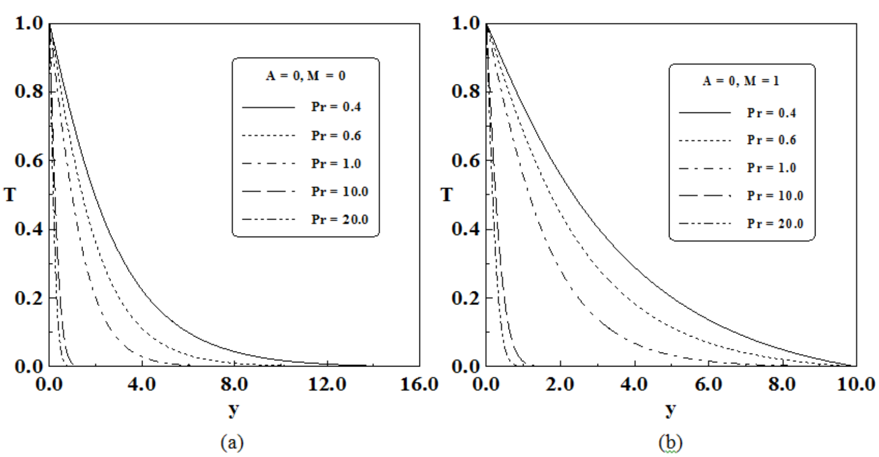

Fig. 8: Temperature profiles over a range of Pr with $A = 0$ at: (a) $M = 0$ (b) $M = 1$

Table 6: Comparison between the values of $G''(0)$ at $A = 0.0$ and $M = 0$ for different values of Pr.

Pr	G″(0)			
	Carragher and Crane [27]	Grubka and Bobba [28]	Pantokratoras [10]	Present work
0.023			-1.0050	-0.9999893
0.100			-1.0050	-0.9999893
0.700	-1.0		-1.0050	-0.9999893
1.000			-1.0050	-0.9999893
10.00		-1.0	-1.0050	-0.9999893

Grubka and Bobba [28] and Pantokratoras [10]. The results are in very good agreement, Table 4.

4.5 Wall Shear Stress

The dimensionless wall shear stress $G''(0)$ (skin friction) is computed for different values of the Hartmann number and the viscosity parameter A at $\Pr = 1.0$. As seen from Table 5, the absolute value of the dimensionless wall shear stress $|G''(0)|$ increases as A increases which is consistent with the fact that, there is progressive thinning of the boundary layer with increasing A. . Also, the absolute value of the dimensionless wall shear stress $|G''(0)|$ increases as M increases for fixed value of viscosity parameter A, Table 5.

The computed values of $G''(0)$ are compared with those obtained by Carragher and Crane [27], Grubka and Bobba [28] and Pantokratoras [10] The results are in very good agreement, Table 6.

5 Conclusion

The steady two-dimensional incompressible Magnetohydrodynamic (MHD) boundary layer flow of variable viscosity over a heated stretching sheet in the presence of uniform transverse magnetic field has been investigated. The system of non-linear partial differential equations is solved using Lie-group method. The resulting ordinary differential equations are solved numerically using the shooting method coupled with Runge-Kutta scheme. The influences of the Hartmann number M, the viscosity parameter A, and the Prandtl number Pr on the horizontal velocity u/x, vertical velocity v, temperature profiles T, wall heat transfer $-T'(0)$ and the wall shear stress $G''(0)$ (skin friction) were examined. Particular cases of our results are compared with those of Pantokratoras [10], Chiam [26], Carragher and Crane [27], Grubka and Bobba [28] and were found a good agreement with their results. Comments on the work of Mukhopadhyay et al. [4] and Pantokratoras [10] were discussed and corrected.

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