Applied Mathematics and Computation 219 (2013) 11365–11375

Contents lists available at SciVerse ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

New solutions for solving problem of particle trajectories in linear deep-water waves via Lie-group method

Mina B. Abd-el-Malek [a,*], Nagwa A. Badran [a], Hossam S. Hassan [b], Heba H. Abbas [b]

[a] Department of Engineering Mathematics and Physics, Faculty of Engineering, Alexandria University, Alexandria 21544, Egypt
[b] Department of Basic and Applied Science, Arab Academy for Science, Technology and Maritime Transport, P.O. Box 1029, Alexandria, Egypt

ARTICLE INFO

Keywords:
Deep-water waves
Euler's equation
Similarity solutions
Lie group

ABSTRACT

The nonlinear equations of the two-dimensional inviscid incompressible fluid in a constant gravitational field describing the wave propagation on the water surface are considered. The Lie-group method has been applied for determining symmetry reductions of the system of partial differential equations. Lie-group method starts out with a general infinitesimal group of transformations under which the given partial differential equations are invariant. The determining equations are a set of linear differential equations, the solution of which gives the transformation function or the infinitesimals of the dependent and independent variables. After the group has been determined, a solution to the given partial differential equations may be found from the invariant surface condition such that its solution leads to similarity variables that reduce the number of independent variables of the system. Effects of the wavelength λ and time t on the particle path have been studied and the results are plotted.

1. Introduction

As a surface wave passes over deep water, the surface water particles trace roughly circular orbits with a diameter equal to the height of the wave. As the wave crest approaches, the surface water particles rise with respect to the still-water level and move forward, while when the crest passes, the particles begin to fall, [1]. The classical description of these particle paths is obtained within the framework of linear water wave theory [2–9]: all water particles trace a circular orbit, the diameter of which decreases with depth so that the orbital motion practically ceases at depth equal to one-half the wavelength. These features have important practical consequences. For example, a submarine at a depth below half a wavelength would hardly notice the motion of the surface wave, for this reason submarines dive during storms in the open sea, [1]. Garabedian [10] considered the symmetry property of periodic waves of finite depth with a variational approach provided that each streamline has a single crest and a single trough per wavelength except for the flat bottom. Matioc [11] determined the phase portrait of a Hamiltonian system of equations describing the motion of the particles in linear water waves. He showed that for linear water wave no particle trajectory is closed, unless the free surface is flat. Each trajectory involves over a period a backward/forward movement of the particle, and the path is an elliptical arc (which degenerates on the flat bed) but with a forward drift. Constantin et al. [12] showed that any critical point of a functional representing the total energy of the flow adjusted with a measure of the vorticity, subject to the constraints of fixed mass and horizontal momentum, is a steady water wave. They assumed there is no surface tension. In the presence of surface tension, there are other variational principles for steady water waves that employ the spatial dynamics method, which treats the horizontal spatial variable as the

* Corresponding author. Current address: Department of Mathematics and Actuarial Science, The American University in Cairo, Egypt.
E-mail addresses: minab@aucegypt.edu (M.B. Abd-el-Malek), hossams@aast.edu (H.S. Hassan), eng.hebahussien@hotmail.com (H.H. Abbas).

0096-3003/$ - see front matter
http://dx.doi.org/10.1016/j.amc.2013.05.059

dynamic variable. The existence proofs that use these principles require non-vanishing surface tension. Gerstner [13] constructed an explicit example of a periodic traveling wave in water of infinite depth with a particular nonzero vorticity. Constantin and Escher [14] used an appropriate hodograph change of variable that transforms the free boundary problem (corresponding in a frame moving at the constant wave speed to the governing equations for water waves with vorticity) into a nonlinear boundary problem for a quasi-linear elliptic equation in a fixed rectangular domain. They proved that the profile of a periodic traveling wave propagating at the surface of water with a flat bed in a flow with Hölder continuously differentiable vorticity and without stagnation points must be real analytic if it is Hölder continuously differentiable. Constantin and Strauss [15] considered the classical water wave problem described by the Euler equations with a free surface under the influence of gravity over a flat bottom. They constructed two-dimensional inviscid periodic traveling waves with vorticity. They are symmetric waves whose profiles are monotone between each crest and trough. They used bifurcation and degree theory to construct a global connected set of such solutions. Constantin et al. [1] used phase plane considerations for the explicit nonlinear system available withinlinear water wave theory. They showed that within the framework of linear water wave theory the particle paths in a deep-water wave are not closed: there is a forward drift over a period, which decreases with greater depth. Hur [16] used a quasi-linear elliptic boundary value problem in which the vorticity function appears only as a part of coefficient functions, and thus it is free from restrictions on the vorticity. He proved that for arbitrary vorticities, periodic water waves of finite depth are symmetric and monotone if their profile has a single minimum (trough) per wavelength near which it is monotone and every streamline attains a minimum below the trough. Toland [17] proved the symmetry of periodic waves in case of zero vorticity under the assumption that every streamline has a maximum and a minimum per wavelength except for the flat bottom.

In this work, Lie-group method is applied to the nonlinear equations of the two-dimensional inviscid incompressible fluid in a constant gravitational field which describe the wave propagation on the water surface, for determining symmetry reductions of the given partial differential equation, [18–29]. The resulting system of nonlinear differential equations is then solved using MATLAP package to get the particle path.

2. Mathematical formulation of the problem

The governing equations for the propagation of two-dimensional gravity deep-water waves in Cartesian coordinates (x, y) with neglecting viscosity and assuming constant density are:

Continuity equation:

$$\frac{\partial \bar{u}}{\partial \bar{x}} + \frac{\partial \bar{v}}{\partial \bar{y}} = 0. \tag{2.1}$$

Euler's equations:

$$\frac{\partial \bar{u}}{\partial \bar{t}} + \bar{u}\frac{\partial \bar{u}}{\partial \bar{x}} + \bar{v}\frac{\partial \bar{u}}{\partial \bar{y}} = -\frac{\partial \bar{P}}{\partial \bar{x}}, \tag{2.2}$$

$$\frac{\partial \bar{v}}{\partial \bar{t}} + \bar{u}\frac{\partial \bar{v}}{\partial \bar{x}} + \bar{v}\frac{\partial \bar{v}}{\partial \bar{y}} = -\frac{\partial \bar{P}}{\partial \bar{y}} - g, \tag{2.3}$$

where $\bar{u}\ (\bar{x}, \bar{y}, \bar{t})$ and $\bar{v}\ (\bar{x}, \bar{y}, \bar{t})$ are the velocity components in the direction of $\bar{x}$ and $\bar{y}$ respectively, $\bar{P}\ (\bar{x}, \bar{y}, \bar{t})$ denotes the pressure and g is the gravitational constant of acceleration. The horizontal x-axis is in the direction of wave propagation, the y-axis pointing vertically upwards and the origin lies at the mean water level, Fig. 1. The only external force acting on the water is gravity.

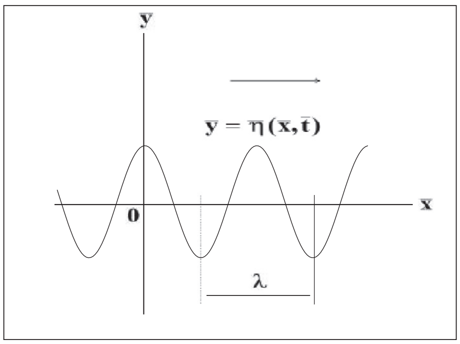

Fig. 1. A deep-water wave with wavelength λ.

The boundary conditions are

$$\text{(i) } \overline{P} = P_0, \quad \text{on } \overline{y} = \overline{\eta}(\overline{x}, \overline{t}), \tag{2.4}$$

$$\text{(ii) } \overline{v} = \frac{\partial \overline{\eta}}{\partial \overline{t}} + \overline{u}\frac{\partial \overline{\eta}}{\partial \overline{x}}, \quad \text{on } \overline{y} = \overline{\eta}(\overline{x}, \overline{t}), \tag{2.5}$$

$$\text{(iii) } (\overline{u}, \overline{v}) \to (0,0), \quad \text{as } \overline{y} \to -\infty, \tag{2.6}$$

where P_0 is the constant atmospheric pressure.

An important category of flows, [1], are those of zero vorticity, characterized by the additional assumption, which is

$$\frac{\partial \overline{u}}{\partial \overline{y}} = \frac{\partial \overline{v}}{\partial \overline{x}}. \tag{2.7}$$

The vorticity of a flow measures the local spin or rotation of a fluid element. In flows for which (2.7) holds the local whirl is completely absent and for this reason such flows are called irrotational, [1]. Experimental evidence indicates that for waves entering a region of still water the assumption of irrotational flow is realistic [5] and, as a consequence of Kelvin's circulation theorem, a water flow that is irrotational initially has to be irrotational at all later times. It is thus reasonable to consider that water motions starting from rest will remain irrotational at later times.

Let

$$\overline{P} = P_0 - g\overline{y} + g\overline{p}, \tag{2.8}$$

where $\overline{p}$ measuring the deviation from the hydrostatic pressure distribution.

The variables in Eqs. (2.1)–(2.8) are dimensionless according to

$$x = \frac{\overline{x}}{\lambda}, \quad y = \frac{\overline{y}}{\lambda}, \quad t = \sqrt{\frac{g}{\lambda}}\overline{t}, \quad u = \frac{\overline{u}}{\sqrt{g\lambda}}, \quad v = \frac{\overline{v}}{\sqrt{g\lambda}}, \quad \eta = \frac{\overline{\eta}}{\lambda}, \quad p = \frac{\overline{p}}{\lambda}, \tag{2.9}$$

where λ is the wavelength.

Substitution from (2.8) and (2.9) into (2.1)–(2.3) yields

$$u_x + v_y = 0, \tag{2.10}$$

$$u_t + uu_x + vu_y = -p_x, \tag{2.11}$$

$$v_t + uv_x + vv_y = -p_y, \tag{2.12}$$

where subscripts denote partial derivatives with respect to the indicated variable.

The boundary conditions (2.4)–(2.6) will be

$$\text{(i) } p = y, \quad \text{on } y = \eta(x,t), \tag{2.13}$$

$$\text{(ii) } v = \eta_t + u\eta_x, \quad \text{on } y = \eta(x,t), \tag{2.14}$$

$$\text{(iii) } (u, v) \to (0,0), \quad \text{as } y \to -\infty. \tag{2.15}$$

Eq. (2.7) will be

$$u_y = v_x, \tag{2.16}$$

From the Continuity Eq. (2.10), there exists a dimensional stream function $\psi(x,y,t)$ such that

$$u(x,y,t) = \frac{\partial \psi(x,y,t)}{\partial y}, \quad v(x,y,t) = -\frac{\partial \psi(x,y,t)}{\partial x}, \tag{2.17}$$

which satisfies Eq. (2.10) identically.

Substitution from (2.17) into (2.11) and (2.12) yields

$$\psi_{yt} + \psi_y\psi_{xy} - \psi_x\psi_{yy} = -p_x, \tag{2.18}$$

$$-\psi_{xt} - \psi_y\psi_{xx} + \psi_x\psi_{xy} = -p_y. \tag{2.19}$$

The boundary conditions (2.13)–(2.15) will be

$$\text{(i) } p = y, \quad \text{on } y = \eta(x,t), \tag{2.20}$$

$$\text{(ii) } \psi_x = -\eta_t - \psi_y\eta_x, \quad \text{on } y = \eta(x,t), \tag{2.21}$$

$$\text{(iii)}\ (\psi_y, \psi_x) \to (0, 0), \quad \text{as } y \to -\infty. \tag{2.22}$$

Eq. (2.16) reduces to

$$\psi_{xx} + \psi_{yy} = 0. \tag{2.23}$$

3. Solution of the problem

At first, we derive the similarity solutions using Lie-group method under which (2.18)–(2.23) are invariant, and then we use these symmetries to determine the similarity variables.

3.1. Lie point symmetries

Consider the one-parameter (ε) Lie group of infinitesimal transformations in $(x,y,t;\psi,p)$ given by

$$\left.\begin{aligned} x^* &= x + \varepsilon X(x,y,t;\psi,p) + O(\varepsilon^2),\\ y^* &= y + \varepsilon Y(x,y,t;\psi,p) + O(\varepsilon^2),\\ t^* &= t + \varepsilon T(x,y,t;\psi,p) + O(\varepsilon^2),\\ \psi^* &= \psi + \varepsilon \Psi(x,y,t;\psi,p) + O(\varepsilon^2),\\ p^* &= p + \varepsilon H(x,y,t;\psi,p) + O(\varepsilon^2), \end{aligned}\right\} \tag{3.1}$$

where "ε" is the group parameter.

The partial differential equations (2.18) and (2.19) are said to admit a symmetry generated by the vector field

$$\Gamma \equiv X\frac{\partial}{\partial x} + Y\frac{\partial}{\partial y} + T\frac{\partial}{\partial t} + \Psi\frac{\partial}{\partial \psi} + H\frac{\partial}{\partial p}, \tag{3.2}$$

if they are left invariant by the transformation $(x,y,t;\psi,p) \to (x^*,y^*,t^*;\psi^*,p^*)$.

The solutions $\psi = \psi(x,y,t)$ and $p = p(x,y,t)$ are invariant under the symmetry (3.2) if

$$\Phi_\psi = \Gamma(\psi - \psi(x,y,t)) = 0, \text{ when } \psi = \psi(x,y,t) \tag{3.3}$$

and

$$\Phi_p = \Gamma(p - p(x,y,t)) = 0, \text{ when } p = p(x,y,t). \tag{3.4}$$

Assume,

$$\Delta_1 = \psi_{yt} + \psi_y\psi_{xy} - \psi_x\psi_{yy} + p_x, \tag{3.5}$$

$$\Delta_2 = -\psi_{xt} - \psi_y\psi_{xx} + \psi_x\psi_{xy} + p_y. \tag{3.6}$$

A vector Γ given by (3.2), is said to be a Lie point symmetry vector field for (2.18) and (2.19) if

$$\Gamma^{[2]}(\Delta_i)|_{\Delta_i=0} = 0, \quad i = 1, 2, \tag{3.7}$$

where

$$\begin{aligned}\Gamma^{[2]} \equiv\ & X\frac{\partial}{\partial x} + Y\frac{\partial}{\partial y} + T\frac{\partial}{\partial t} + \Psi\frac{\partial}{\partial \psi} + H\frac{\partial}{\partial p} + \Psi^x\frac{\partial}{\partial \psi_x} + \Psi^y\frac{\partial}{\partial \psi_y} + H^x\frac{\partial}{\partial p_x} + H^y\frac{\partial}{\partial p_y} + \Psi^{xt}\frac{\partial}{\partial \psi_{xt}} + \Psi^{xy}\frac{\partial}{\partial \psi_{xy}} + \Psi^{yt}\frac{\partial}{\partial \psi_{yt}}\\ & + \Psi^{xx}\frac{\partial}{\partial \psi_{xx}} + \Psi^{yy}\frac{\partial}{\partial \psi_{yy}}\end{aligned} \tag{3.8}$$

is the second prolongation of Γ.

To calculate the prolongation of the given transformation, we need to differentiate (3.1) with respect to each of the variables x, y and t. To do this, we introduce the following total derivatives

$$\left.\begin{aligned} D_x &\equiv \partial_x + \psi_x\partial_\psi + p_x\partial_p + \psi_{xx}\partial_{\psi_x} + p_{xx}\partial_{p_x} + \psi_{xy}\partial_{\psi_y} + \cdots\cdots\cdots,\\ D_y &\equiv \partial_y + \psi_y\partial_\psi + p_y\partial_p + \psi_{yy}\partial_{\psi_y} + p_{yy}\partial_{p_y} + \psi_{xy}\partial_{\psi_x} + \cdots\cdots\cdots,\\ D_t &\equiv \partial_t + \psi_t\partial_\psi + p_t\partial_p + \psi_{tt}\partial_{\psi_t} + p_{tt}\partial_{p_t} + \psi_{xt}\partial_{\psi_x} + \cdots\cdots\cdots \end{aligned}\right\} \tag{3.9}$$

Eq. (3.7) gives the following system of linear partial differential equations

$$\psi_{xy}\Psi^y - \psi_{yy}\Psi^x + H^x + \Psi^{yt} + \psi_y\Psi^{xy} - \psi_x\Psi^{yy} = 0, \tag{3.10}$$

$$\Psi^{xt} + \psi_{xx}\Psi^y + \psi_y\Psi^{xx} - \psi_{xy}\Psi^x - \psi_x\Psi^{xy} - H^y = 0. \tag{3.11}$$

The components Ψ^x, Ψ^y, H^x, H^y, Ψ^{xx}, Ψ^{xy}, Ψ^{xt}, Ψ^{yy} and Ψ^{yt} can be determined from the following expressions

$$\left.\begin{aligned} &\Psi^S = D_S\Psi - \psi_x D_S X - \psi_y D_S Y - \psi_t D_S T,\\ &H^N = D_N H - p_x D_N X - p_y D_N Y - p_t D_N T,\\ &\Psi^{jS} = D_S\Psi^j - \psi_{jx} D_S X - \psi_{jy} D_S Y - \psi_{jt} D_S T, \end{aligned}\right\} \tag{3.12}$$

where S, j stands for x, y, t and N stands for x, y.

Substitution from (3.12) into (3.10) will lead to a large expression, then, equating to zero the coefficients of $\psi_x\psi_y\psi_{yyy},\ \psi_x^2\psi_y\psi_{yyy},\ \psi_x\psi_y^2\psi_{yyy},\ \psi_x\psi_y\psi_t\psi_{yyy},\ \psi_y^2\psi_{xy},\ \psi_x\psi_y\psi_{xy},\ \psi_y\psi_{tt}$ and p_t, gives

$$\Psi_p = X_p = Y_p = T_p = Y_\psi = X_\psi = T_\psi = T_x = 0. \tag{3.13}$$

Substitution from (3.13) into (3.10) will remove many terms. Then, equating to zero the coefficients of the derivatives of the dependent variables, leads to the following system of determining equations:

$$\left.\begin{aligned} &\Psi_{\psi\psi} = 0, \quad \Psi_{y\psi} = 0, \quad \Psi_{x\psi} = 0,\\ &X_{yy} = 0, \quad Y_{xy} = 0, \quad T_x = T_y = 0,\\ &Y_x = -X_y, \quad X_t = \Psi_y,\\ &Y_t = -\Psi_x, \quad Y_{yy} = X_{xy},\\ &\Psi_{\psi t} - Y_{yt} + \Psi_{xy} = 0,\\ &H_x + \Psi_{yt} = 0,\\ &H_\psi - X_{yt} - \Psi_{yy} = 0,\\ &H_p - X_x - \Psi_\psi + Y_y + T_t = 0,\\ &H_p - 2\Psi_\psi + 2Y_y = 0. \end{aligned}\right\} \tag{3.14}$$

Substitution from (3.12) into (3.11) will lead to the following system of determining equations:

$$\left.\begin{aligned} &X_{xy} = 0, \quad X_{xx} = 0, \quad \Psi_{xx} = Y_{xt} + H_\psi,\\ &\Psi_{\psi t} = \Psi_{xy} + X_{xt}, \quad \Psi_{xt} = H_y,\\ &H_p + X_x - \Psi_\psi - Y_y + T_t = 0, \quad 2X_x - 2\Psi_\psi + H_p = 0. \end{aligned}\right\} \tag{3.15}$$

Solving the system of determining Eqs. (3.14) and (3.15), in view of the invariance of the initial and boundary conditions (2.20)–(2.23), and assume $\eta = \cos(2\pi(x - ct))$, where $c = \sqrt{\frac{\lambda}{2\pi}}$, yields

$$X = cc_1, \quad Y = c_2, \quad T = c_1, \quad \Psi = c_3 t + c_4, \quad H = c_2. \tag{3.16}$$

So, the nonlinear Eqs. (2.18) and (2.19) admit the four-parameter Lie group of point symmetries generated by

$$\Gamma_1 \equiv c\frac{\partial}{\partial x} + \frac{\partial}{\partial t}, \quad \Gamma_2 \equiv \frac{\partial}{\partial y} + \frac{\partial}{\partial p}, \quad \Gamma_3 \equiv t\frac{\partial}{\partial \psi}, \quad \Gamma_4 \equiv \frac{\partial}{\partial \psi}. \tag{3.17}$$

The only non-zero commutation relation of these symmetries is $[\Gamma_1, \Gamma_3] = \Gamma_4$, where $[\Gamma_i, \Gamma_j] = \Gamma_i\Gamma_j - \Gamma_j\Gamma_i$. In the classification presented by both Mubarakzyanov [30] and Patera and Winternitz [31], this algebra is $A_{3.1} \oplus A_1$.

The finite transformations corresponding to the symmetries Γ_1, Γ_2, Γ_3 and Γ_4 are respectively

$$\begin{array}{llllll} \Gamma_1: & x^* = x + \varepsilon_1 c, & y^* = y, & t^* = t + \varepsilon_1, & \psi^* = \psi, & p^* = p,\\ \Gamma_2: & x^* = x, & y^* = y + \varepsilon_2, & t^* = t, & \psi^* = \psi, & p^* = p + \varepsilon_2,\\ \Gamma_3: & x^* = x, & y^* = y, & t^* = t, & \psi^* = \psi + \varepsilon_3 t, & p^* = p,\\ \Gamma_4: & x^* = x, & y^* = y, & t^* = t, & \psi^* = \psi + \varepsilon_4, & P^* = P, \end{array} \tag{3.18}$$

where ε_1, ε_2, ε_3 and ε_4 are the group parameters.

3.2. One-dimensional optimal system of subalgebras of the symmetry group

Since the symmetry Lie algebra is four-dimensional which given by the operators (3.17), we look for solutions invariant under the linear combination of these operators. All the possible invariant solutions can be obtained by determine the optimal system of one-dimensional subalgebras of the given system of partial differential equations. This is the most systematic procedure, [22]. Following Olver's approach given in [22] by firstly computing the commutators of the symmetry Lie algebra (3.17), which is obtained in Section 3.1, and thereafter obtaining the adjoint representations. The adjoint action on Lie algebras is defined by the adjoint operator given by

$$Ad_{\exp(a\Gamma_i)}\langle\Gamma_j\rangle = e^{-a\Gamma_i}\Gamma_j e^{a\Gamma_i}, \tag{3.19}$$

where, "a" is a small parameter.

This operator can be rewritten in terms of Lie brackets using Campbell–Baker–Hausdorff theorem [32] as

$$Ad_{\exp(a\Gamma_i)}\langle\Gamma_j\rangle = \Gamma_j - a[\Gamma_i,\Gamma_j] + \frac{a^2}{2!}[\Gamma_i,[\Gamma_i,\Gamma_j]] - \cdots\cdots\cdots \tag{3.20}$$

For our problem, $\Omega = \langle\Gamma_1,\Gamma_2,\Gamma_3,\Gamma_4\rangle$ is the Lie algebra associated with the symmetry group. The nontrivial adjoint actions of Lie symmetry algebra Ω are $Ad_{\exp(a\Gamma_1)}\langle\Gamma_3\rangle = \Gamma_3 - a\Gamma_4$ and $Ad_{\exp(a\Gamma_3)}\langle\Gamma_1\rangle = \Gamma_1 + a\Gamma_4$.

To construct the one-dimensional optimal system of Ω, consider a general element of Ω given by

$$E = a_1\Gamma_1 + a_2\Gamma_2 + a_3\Gamma_3 + a_4\Gamma_4 \tag{3.21}$$

for some constants a_1, a_2, a_3 and a_4, and probe whether E can be transformed to a new element E' under the general adjoint action, where E' takes a simpler form than E, [33].

Let,

$$E' = Ad_{\exp(a\Gamma_i)}\langle E\rangle = a_1'\Gamma_1 + a_2'\Gamma_2 + a_3'\Gamma_3 + a_4'\Gamma_4. \tag{3.22}$$

We make appropriate choice of a such that the a_i''s can be made 0 or 1. We end up with simpler forms of E that will constitute the one-dimensional optimal system.

By substitution $\Gamma_i = \Gamma_1$ in (3.22) and dropping the primes, we get

$$E' = a_1\Gamma_1 + a_2\Gamma_2 + a_3\Gamma_3 + (a_4 - aa_3)\Gamma_4. \tag{3.23}$$

Now, Eq. (3.23) prompts the consideration of the cases $a_3 \neq 0$ and $a_3 = 0$.

Case (1): $a_3 \neq 0$.

By choosing ($a = a_4/a_3$) and scaling the resulting operator by a_3, Eq. (3.23) will be

$$E' = a_1\Gamma_1 + a_2\Gamma_2 + \Gamma_3. \tag{3.24}$$

We can further consider the subcases $a_1,a_2 \neq 0$, $a_1 = 0$, $a_2 = 0$ and $a_1 = a_2 = 0$. Therefore, an optimal system of one-dimensional subalgebra for this case is given by $\{\Gamma_1 + \beta\Gamma_2 + \sigma\Gamma_3, \Gamma_2 + \beta\Gamma_3, \Gamma_1 + \beta\Gamma_3, \Gamma_3\}$, where, $\beta \in R$ and $\sigma \in R$.

Case (2): $a_3 = 0$.

Using repeatedly the adjoint operation to simplify E, an optimal system of one-dimensional subalgebra for this case is given by $\{\Gamma_1, \Gamma_2, \Gamma_4, \Gamma_1 + \beta\Gamma_2, \Gamma_2 + \beta\Gamma_4\}$.

In summary, the optimal system of one-dimensional subalgebras of the symmetry Lie algebra is

$$\Theta = \{\Gamma_1,\Gamma_2,\Gamma_3,\Gamma_4,\Gamma_1+\beta\Gamma_2,\Gamma_1+\beta\Gamma_3,\Gamma_2+\beta\Gamma_3,\Gamma_2+\beta\Gamma_4,\Gamma_1+\beta\Gamma_2+\sigma\Gamma_3\}. \tag{3.25}$$

Table 1 shows the solution of the invariant surface conditions associated with the optimal system.

For Γ_1, the characteristic

$$\Phi = (\Phi_\psi, \Phi_p) \tag{3.26}$$

has the components

$$\Phi_\psi = c\psi_x + \psi_t, \quad \Phi_p = cp_x + p_t. \tag{3.27}$$

Therefore, the general solutions of the invariant surface conditions (3.3) and (3.4) are

$$\psi = f(y)L(\alpha), \quad p = N(y)K(\alpha), \quad \alpha = 2\pi(x - ct). \tag{3.28}$$

Substitution from (3.28) into (2.17) yields

$$u = L(\alpha)\frac{df}{dy}, \quad v = -2\pi f(y)\frac{dL}{d\alpha}. \tag{3.29}$$

Table 1
Solutions of the invariant surface conditions associated with the optimal system.

Generator	Characteristic $\Phi = (\Phi_\psi, \Phi_p)$	Solution of the invariant surface conditions
Γ_1	$\Phi_\psi = c\psi_x + \psi_t$, $\Phi_p = c\,p_x + p_t$	$\psi = f(y)L(\alpha)$, $p = N(y)K(\alpha)$, $\alpha = 2\pi(x - c\,t)$
Γ_2	$\Phi_\psi = \psi_y$, $\Phi_p = 1 - p_y$	$\psi = f_1(x,t)$, $p = y + N_1(x,t)$
Γ_3	$\Phi_\psi = t$, $\Phi_p = 0$	No solution
Γ_4	$\Phi_\psi = 1$, $\Phi_p = 0$	No solution
$\Gamma_1 + \beta\Gamma_2$	$\Phi_\psi = c\psi_x + \psi_t + \beta\psi_y$, $\Phi_p = c\,p_x + p_t + \beta p_y - \beta$	$\psi = f_2(x - c\,t, y - \beta t)$, $p = y + N_2(x - ct, y - \beta t)$
$\Gamma_1 + \beta\Gamma_3$	$\Phi_\psi = -c\psi_x - \psi_t + \beta t$, $\Phi_p = -c\,p_x - p_t$.	$\psi = \frac{\beta t^2}{2} + f(y)L(\alpha),\ p = N(y)K(\alpha),\ \alpha = 2\pi(x - ct)$
$\Gamma_2 + \beta\Gamma_3$	$\Phi_\psi = -\psi_y + \beta t$, $\Phi_p = 1 - p_y$	$\psi = \beta t\,y + f_3(x,t)$, $p = y + N_3(x,t)$
$\Gamma_2 + \beta\Gamma_4$	$\Phi_\psi = -\psi_y + \beta$, $\Phi_p = 1 - p_y$	$\psi = \beta y + f_4(x,t)$, $p = y + N_4(x,t)$
$\Gamma_1 + \beta\Gamma_2 + \sigma\Gamma_3$	$\Phi_\psi = -c\psi_x - \psi_t - \beta\psi_y + \sigma t$, $\Phi_p = -c\,p_x - p_t - \beta p_y + \beta$	$\psi = \frac{\sigma t^2}{2} + f_5(x - ct, y - \beta t),\ p = \beta t + N_5(x - ct, y - \beta t)$

Substitution from (3.29) into (2.23) yields

$$u = 2\pi c\ e^{2\pi y}\cos[2\pi(x-ct)],\quad v = 2\pi c\ e^{2\pi y}\sin[2\pi(x-ct)]. \tag{3.30}$$

Substitution from (3.30) into (2.11) and (2.12) yields

$$p = k\ c\lambda\ e^{ky}\cos[2\pi(x-ct)] - 2\pi^2\ c^2\ e^{2ky} - g\ y + h(t). \tag{3.31}$$

By recovering the physical variables, Eq. (3.30) will be

$$\begin{aligned} u &= \lambda\omega e^{ky}\cos(kx-\omega t),\\ v &= \lambda\omega e^{ky}\sin(kx-\omega t), \end{aligned} \tag{3.32}$$

where, $k = \frac{2\pi}{\lambda}$ and $\omega = \sqrt{2\pi g}$.

For Γ_2, the characteristic (3.26) has the components

$$\Phi_\psi = \psi_y,\quad \Phi_p = 1 - p_y. \tag{3.33}$$

Therefore, the general solutions of the invariant surface conditions (3.3) and (3.4) are

$$\psi = f_1(x,t),\quad p = y + N_1(x,t). \tag{3.34}$$

Substitution from (3.33) into (2.17) yields

$$u = 0,\quad v = -\frac{\partial f_1}{\partial x}. \tag{3.35}$$

Eq. (3.35) is a solution of the continuity Eq. (2.10) and the equations of motion (2.11) and (2.12), even though it is not a particularly interesting one (no axial velocity) which contradicts the boundary conditions. So, no solutions are invariant under the group generated by Γ_2.

For Γ_3, the characteristic (3.26) has the components

$$\Phi_\psi = t,\quad \Phi_p = 0. \tag{3.36}$$

So, no solutions are invariant under the group generated by Γ_3.

For Γ_4, the characteristic (3.26) has the components

$$\Phi_\psi = 1,\quad \Phi_p = 0. \tag{3.37}$$

So, no solutions are invariant under the group generated by Γ_4.

For $\Gamma_1 + \beta\Gamma_2$, the characteristic (3.26) has the components

$$\Phi_\psi = c\psi_x + \psi_t + \beta\psi_y,\quad \Phi_p = cp_x + p_t + \beta p_y - \beta. \tag{3.38}$$

Therefore, the general solutions of the invariant surface condition (3.3) and (3.4) are

$$\psi = f_2(x-ct, y-\beta t),\quad p = y + N_2(x-ct, y-\beta t). \tag{3.39}$$

Substitution from (3.39) into (2.17) yields

$$u = \frac{\partial f_2}{\partial y},\quad v = -\frac{\partial f_2}{\partial x}. \tag{3.40}$$

Eq. (3.40) is a solution of the continuity Eq. (2.10) and the equations of motion (2.11) and (2.12), even though it is not a particularly interesting one which contradicts the boundary conditions. So, no solutions are invariant under the group generated by $\Gamma_1 + \beta\Gamma_2$.

For $\Gamma_1 + \beta\Gamma_3$, the characteristic (3.26) has the components

$$\Phi_\psi = -c\psi_x - \psi_t + \beta t,\quad \Phi_p = -cp_x - p_t. \tag{3.41}$$

Therefore, the general solutions of the invariant surface conditions (3.3) and (3.4) are

$$\psi = \frac{\beta t^2}{2} + f(y)L(\alpha),\quad p = N(y)K(\alpha),\quad \alpha = 2\pi(x-ct). \tag{3.42}$$

Substitution from (3.42) into (2.17) yields

$$u = L(\alpha)\frac{df}{dy},\quad v = -2\pi f(y)\frac{dL}{d\alpha}, \tag{3.43}$$

which gives the same solution invariant under Γ_1.

For $\Gamma_2 + \beta\Gamma_3$, the characteristic (3.26) has the components

$$\Phi_\psi = -\psi_y + \beta t,\quad \Phi_p = 1 - p_y. \tag{3.44}$$

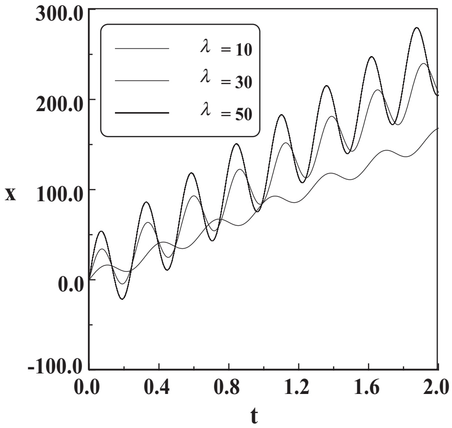

Fig. 2. *x*-Location profiles over a range of λ.

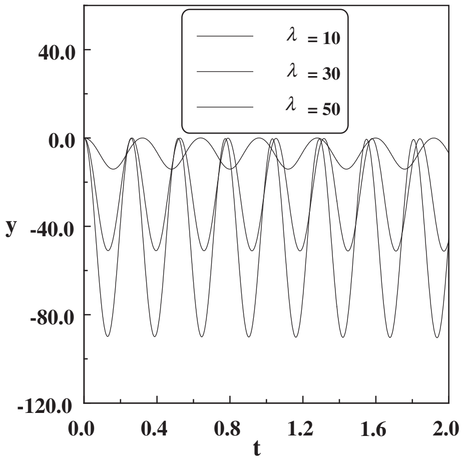

Fig. 3. *y*-Location profiles over a range of λ.

Therefore, the general solutions of the invariant surface conditions (3.3) and (3.4) are

$$\psi = \beta t y + f_3(x,t), \quad p = y + N_3(x,t). \tag{3.45}$$

Substitution from (3.45) into (2.17) yields

$$u = \beta t, \quad v = -\frac{\partial f_3}{\partial x}. \tag{3.46}$$

Eq. (3.46) is a solution of the continuity Eq. (2.10) and the equations of motion (2.11) and (2.12), even though it is not a particularly interesting one which contradicts the boundary conditions. So, no solutions are invariant under the group generated by $\Gamma_2 + \beta\Gamma_3$.

For $\Gamma_2 + \beta\Gamma_4$, the characteristic (3.26) has the components

$$\Phi_\psi = -\psi_y + \beta, \quad \Phi_p = 1 - p_y. \tag{3.47}$$

Therefore, the general solutions of the invariant surface conditions (3.3) and (3.4) are

$$\psi = \beta y + f_4(x,t), \quad p = y + N_4(x,t). \tag{3.48}$$

Substitution from (3.48) into (2.17) yields

$$u = \beta, \quad v = -\frac{\partial f_4}{\partial x}. \tag{3.49}$$

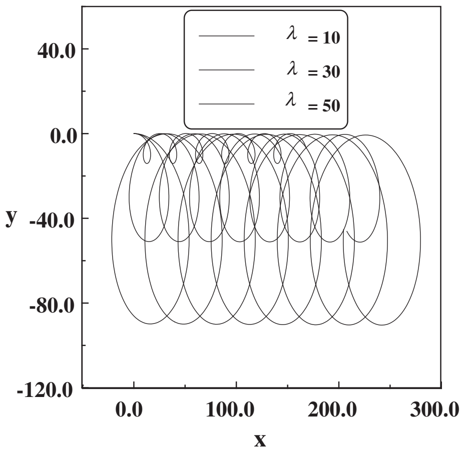

Fig. 4. Particle path profiles over a range of λ at $t = 2$.

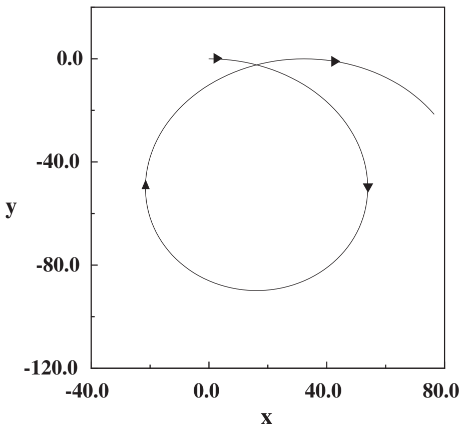

Fig. 5. Particle path profiles at $\lambda = 50$ and $t = 0.3$.

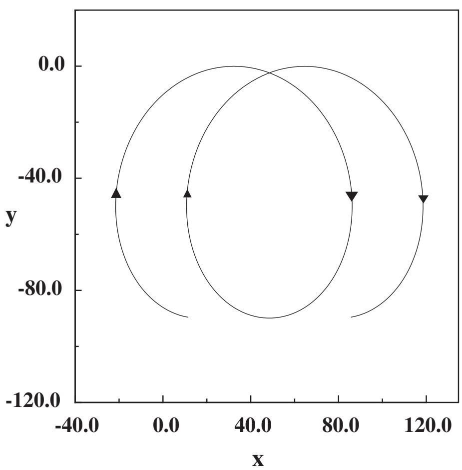

Fig. 6. Particle path profiles at $\lambda = 50$ and $t = 0.63$.

Eq. (3.49) is a solution of the continuity Eq. (2.10) and the equations of motion (2.11) and (2.12), even though it is not a particularly interesting one which contradicts the boundary conditions. So, no solutions are invariant under the group generated by $\Gamma_2 + \beta\Gamma_4$.

For $\Gamma_1 + \beta\Gamma_2 + \sigma\Gamma_3$, the characteristic (3.26) has the components

$$\Phi_\psi = -c\psi_x - \psi_t - \beta\psi_y + \sigma t, \quad \Phi_p = -cp_x - p_t - \beta p_y + \beta. \tag{3.50}$$

Therefore, the general solutions of the invariant surface conditions (3.3) and (3.4) are

$$\psi = \frac{\sigma t^2}{2} + f_5(x - ct, y - \beta t), \quad p = \beta t + N_5(x - ct, y - \beta t). \tag{3.51}$$

Substitution from (3.51) into (2.17) yields

$$u = \frac{\partial f_5}{\partial y}, \quad v = -\frac{\partial f_5}{\partial x}. \tag{3.52}$$

Eq. (3.52) is a solution of the continuity Eq. (2.10) and the equations of motion (2.11) and (2.12), even though it is not a particularly interesting one which contradicts the boundary conditions. So, no solutions are invariant under the group generated by $\Gamma_1 + \beta\Gamma_2 + \sigma\Gamma_3$.

4. Results and discussion

For a particle in the fluid domain, let $(x(t), y(t))$ denote its location as a function of time, where $\frac{dx(t)}{dt} = u$ and $\frac{dy(t)}{dt} = v$. So, from (3.32) we get

$$\dot{x}(t) = \lambda\omega e^{ky}\cos(kx - \omega t) \tag{4.1}$$

and

$$\dot{y}(t) = \lambda\omega e^{ky}\sin(kx - \omega t). \tag{4.2}$$

The nonlinear system of Eqs. (4.1) and (4.2) is solved using MATLAB package to get the particle path.

4.1. Effect of the wavelength λ

Over a range of the wavelength λ, Figs. 2 and 3 illustrate the behavior of the x-location and y-location, respectively, as a function of time. Fig. 4 illustrates the particle path $(x(t), y(t))$ at $t = 2$ and over a range of the wavelength λ. As seen, the particle paths in a deep-water wave are not closed. Each particle trajectory involves over a period a backward/forward movement, and the path is an elliptical arc with a forward drift which agree with the results obtained by Constantin et al. [1]. Also, as λ increases, the elliptical arc increases.

4.2. Effect of the time t

At $\lambda = 50$, Figs. 5 and 6 illustrate the behavior of the particle path $(x(t), y(t))$ at $t = 0.3$ and $t = 0.63$, respectively. As mentioned before, the particle path in a deep-water wave is not closed. Also, the particle moves back and up, then forward (first up, then down) and afterwards backwards and downwards which agree with the results obtained by Constantin et al. [1].

5. Conclusion

We have used Lie-group method to obtain the similarity reductions of the nonlinear equations of the two-dimensional inviscid incompressible fluid in a constant gravitational field which describe the wave propagation on the water surface. By determining the transformation group under which the given partial differential equation and its initial and boundary conditions are invariant, we obtained the invariants and the symmetries of this equation. In turn, we used these invariants and symmetries to determine the similarity variables that reduced the number of independent variables. The resulting differential equation is solved using MATLAP package and the results are plotted. We have studied the effect of the wavelength and the time on the particle path and is compared with those obtained by Constantin et al. [1] and were found to agree very well with their results.

Acknowledgment

The authors would like to express their appreciations for the potential reviewer for his/her valuable comments that improved the paper and enhanced the results.

References

[1] A. Constantin, M. Ehrnström, G. Villari, Particle trajectories in linear deep-water waves, Nonlinear Anal. Real World Appl. 9 (2008) 1336–1344.
[2] D.J. Acheson, Elementary Fluid Dynamics, The Clarendon Press, Oxford University Press, New York, 1990.
[3] G.D. Crapper, Introduction to Water Waves, Ellis Horwood Ltd., Chichester, 1984.
[4] L. Debnath, Nonlinear Water Waves, Academic Press, Boston, MA, 1994.
[5] R.S. Johnson, A Modern Introduction to the Mathematical Theory of Water Waves, Cambridge University Press, Cambridge, 1997.
[6] H. Lamb, Hydrodynamics, Cambridge University Press, Cambridge, 1895.
[7] J. Lighthill, Waves in Fluids, Cambridge University Press, Cambridge, 1978.
[8] A. Sommerfeld, Mechanics of Deformable Bodies, Academic Press, New York, NY, 1950.
[9] J.J. Stoker, Water Waves. The Mathematical Theory with Applications, Interscience Publisher, New York, 1957.
[10] P. Garabedian, Surface waves of finite depth, J. Anal. Math. 14 (1965) 161–169.
[11] A.V. Matioc, On particle trajectories in linear water waves, Vienna, Preprint ESI 2222, 2010.
[12] A. Constantin, D. Sattinger, W. Strauss, Variational formulations for steady water waves with vorticity, J. Fluid. Mech. 548 (2006) 151–163.
[13] F. Gerstner, Theorie der Wellen samt einer daraus abgeleiteten theorie der deichprofile, Ann. Phys. 2 (1809) 412–445.
[14] A. Constantin, J. Escher, Symmetry of steady periodic surface water waves with vorticity, J. Fluid Mech. 498 (2004) 171–181.
[15] A. Constantin, W. Strauss, Exact steady periodic water waves with vorticity, Commun. Pure Appl. Math. 57 (2004) 481–527.
[16] V.M. Hur, Symmetry of steady periodic water waves with vorticity, Philos. Trans. R. Soc. A 365 (2007) 2203–2214.
[17] J.F. Toland, On the symmetry theory for Stokes waves of finite and infinite depth, in: Trends in Applications of Mathematics to Mechanics, Monographs and Surveys in Applied Mathematics, Chapman and Hall/CRC, London, UK, 2000, pp. 207–217.
[18] J.M. Hill, Solution of differential equations by means of one-parameter groups, Pitman Publishing Inc., MA, USA, 1982.
[19] R. Seshadri, T.Y. Na, Group invariance in engineering boundary value problems, Springer-Verlag, New York, 1985.
[20] P.E. Hydon, Symmetry Methods for Differential Equations, CUP, Cambridge, 2000.
[21] G.I. Burde, Expanded Lie group transformations and similarity reductions of differential equations, Proc. Inst. Math. NAS Ukr. 43 (2002) 93–101.
[22] P.J. Olver, Applications of Lie Groups to Differential Equations, Springer-Verlag, New-York, 1986.
[23] Y.Z. Boutros, M.B. Abd-el-Malek, N.A. Badran, H.S. Hassan, Lie-group method for unsteady flows in a semi-infinite expanding or contracting pipe with injection or suction through a porous wall, J. Comput. Appl. Math. 197 (2006) 465–494.
[24] Y.Z. Boutros, M.B. Abd-el-Malek, N.A. Badran, H.S. Hassan, Lie-group method of solution for steady two-dimensional boundary-layer stagnation-point flow towards a heated stretching sheet placed in a porous medium, MECCANICA 41 (6) (2007) 681–691.
[25] Y.Z. Boutros, M.B. Abd-el-Malek, N.A. Badran, H.S. Hassan, Lie-group method solution for two-dimensional viscous flow between slowly expanding or contracting walls with weak permeability, J. Appl. Math. Model. 31 (2007) 1092–1108.
[26] M.B. Abd-el-Malek, N.A. Badran, H.S. Hassan, Lie-group method for predicting water content for immiscible flow of two fluids in a porous medium, Appl. Math. Sci. 1 (24) (2007) 1169–1180.
[27] M.B. Abd-el-Malek, H.S. Hassan, Symmetry analysis for steady boundary-layer stagnation-point flow of Rivlin–Ericksen fluid of second grade subject to suction, Nonlinear Anal. Model. Control 15 (4) (2010) 379–396.
[28] M.B., Abd-el-Malek, H.S., Hassan, Lie group method for solving the problem of fission product behaviour in nuclear fuel, in: Proceedings of the Ninth International Conference of Numerical Analysis and Applied Mathematics, ICNAAM 2011, Halkidiki, Greece, pp. 1367–1371.
[29] M.B., Abd-el-Malek, and H.S., Hassan, Solution of Burgers' equation with time-dependent kinematic viscosity via Lie-group analysis, in: Proceedings of the Fifth International Workshop ‘‘Group Analysis of Differential Equations & Integrable Systems’’, Protaras-Cyprus, 2010, pp. 6–14.
[30] G.M. Mubarakzyanov, On solvable Lie algebras, Izv. Vys. Ucheb. Zaved. Mat. 32 (1) (1963) 114–123 (in Russian).
[31] J. Patera, P. Winternitz, Subalgebras of real three and four dimensional Lie algebras, J. Math. Phys. 18 (7) (1977) 1449–1455.
[32] N. Jacobson, Lie Algebras, Dover, New York, 1979.
[33] C. Wafo Soh, Invariant solutions of the unidirectional flow of an electrically charged power-law non-Newtonian fluid over a flat plate in presence of a transverse magnetic field, Commun. Nonlinear Sci. Numer. Simul. 10 (2005) 537–548.