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New solutions for solving Boussinesq equation via potential symmetries method

Mina B. Abd-el-Malek [a,*], Nagwa A. Badran [a], Hossam S. Hassan [b], Heba H. Abbas [b]

[a] Department of Engineering Mathematics and Physics, Faculty of Engineering, Alexandria University, Alexandria 21544, Egypt
[b] Department of Basic and Applied Science, Arab Academy for Science, Technology and Maritime Transport, P.O. Box 1029, Alexandria, Egypt

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ABSTRACT

This work deals with the Boussinesq equation that describes the propagation of the solitary waves with small amplitude on the surface of shallow water. Firstly, the equation is written in a conserved form, a potential function is then assumed reducing it to a system of partial differential equations. The Lie-group method has been applied for determining symmetry reductions of the system of partial differential equations. The solution of the problem by means of Lie-group method reduces the number of independent variables in the given partial differential equation by one leading to nonlinear ordinary differential equations. The resulting non-linear ordinary differential equations are then solved numerically using MATLAP package.

1. Introduction

The Boussinesq equation is widely used in coastal and ocean engineering. One example among others is tsunami wave modeling. These equations can also be used to model tidal oscillations. Of course, these types of wave motion are perfectly described by the Navier–Stokes equations, but currently it is impossible to solve fully three-dimensional (3D) models in any significant domain. Thus, approximate models such as the Boussinesq equations must be used. Several members of the Boussinesq system have been studied in the past, including the classical Boussinesq system.

The existence of solutions for the Boussinesq system of equations had been obtained by Schonbek [1]. The exact solution of the classical Boussinesq equation had been presented by Krishan [2].

The numerical solution of the good Boussinesq equation had been expressed by Manornajan et al. [3] by using Galerkin Methods.

An exact traveling-wave solution of Boussinesq systems was presented by Chen [4]. It was found that it is sufficient to find a solution of an ordinary differential equation and by solving a system of nonlinear algebraic equation it was found the solution of the ordinary differential equation in a prescribed form.

Prabir et al. [5] derived a class of model equation that described the bi-directional propagation of small amplitude long waves on the surface of shallow water. The traveling solitary wave solutions are explicitly constructed for a class of lower order Boussinesq equation of higher-order. The existence and uniqueness of the solution to the Cauchy problem for a class of Boussinesq equation had been investigated by Wang et al. [6].

* Corresponding author at: Department of Mathematics and Actuarial Science, The American University in Cairo, Egypt.
E-mail addresses: minab@aucegypt.edu (M.B. Abd-el-Malek), hossams@aast.edu (H.S. Hassan), eng.hebahussien@hotmail.com (H.H. Abbas).

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A set of two-dimensional Boussinesq equations had been presented by Peregrine [7] which is the basis of much of the modern-day work. The model is invalid in case of deeper water rendering as their linearised dispersion characteristics rapidly diverge from the true behavior. This system of equations is limited to very shallow water.

Beji and Nadaoka [8] have produced an extended Boussinesq system, valid in a variable depth environment, by a simple algebraic manipulation of Peregrine's original system.

Bruzón and Gandarias [9] made a full analysis of a family of Boussinesq equations which include nonlinear dispersion by using the classical Lie method of infinitesimals. In their paper, they considered traveling wave reductions and presented some explicit solutions: solitons and compactons. They derived nonclassical and potential symmetries and proved that the nonclassical method applied to these equations leads to new symmetries, which cannot be obtained by Lie classical method.

Clarkson and Priestley [10] classified the generalized Boussinesq equation by the types of classical and nonclassical symmetry reduction it possesses. They studied the ordinary differential equation arising from their symmetry reductions to see whether they are of Painlevé -type. By virtue of the Painlevé conjecture they concluded that only the Boussinesq equation in the class of equations they have studied may be solvable by inverse scattering.

Some new similarity solutions of the modified Boussinesq equation are presented by Clarkson [11]. These new similarity solutions include reductions to the second and fourth Painlevé equations which are not obtainable using the standard Lie group method for finding group-invariant solutions of partial differential equations; they are determined using a new and direct method which involves no group theoretical techniques.

Clarkson and Ludlow [12] presented a new nonclassical symmetry reductions and exact solutions for a generalized Boussinesq equation. These symmetry reductions are obtained using the direct method, originally developed by Clarkson and Kruskal to study symmetry reductions of the Boussinesq equation, which involves no group theoretic techniques, and using these reductions, they obtained exact solutions expressible in terms of solutions of the second and fourth Painlevé equations, Jacobi, weierstrass elliptic functions and elementary functions.

Gandarias and Bruzón [13] applied the Lie-group formalism and the nonclassical method due to Bluman and Cole to deduce symmetries of the generalized Boussinesq equation, which has the classical Boussinesq equation as a special case. They studied the class of functions for which these equations admit either the classical or the nonclassical method. The reductions obtained are derived. Some new exact solutions are derived.

Kiraz [14] studied generalized Boussinesq equation reduced to previously unknown target ordinary differential equation by applying the extended Lie group transformation and similarity reduction. He obtained target ordinary equation and used it to find the exact solution of generalized Boussinesq equation.

Lockington et al. [15] concluded that, the similarity transforms of the Boussinesq equation in a semi-infinite medium are available when the boundary conditions are a power of time and the Boussinesq equation is reduced from a partial differential equation to a boundary-value problem.

Schäffer and Madsen [16] have suggested that a further set of extended Boussinesq equations involving an additional free parameter can be derived from [17] and these equation systems are all equivalent.

Peregrine [18] presented probably the first finite difference method for a Boussinesq-type equation system. However Abbot et al. [19,20] presented the finite difference solution of the original Boussinesq system for practical engineering problems, and to ensure accurate solutions they analyzed the methods carefully.

Clarkson and Kruskal [21] studied some new similarity reductions of the Boussinesq equation. These new similarity reductions, including some new reductions to the first, second, and fourth Painlevé equations, are determined using a new and direct method that involves no group theoretical techniques.

Levit and Winternitz [22] showed how a specific class of conditional symmetries can be used to reduce a partial differential equation to an ordinary one. In particular, for the Boussinesq equation, these conditional symmetries, together with the ordinary ones, provide all possible reductions to ordinary differential equations.

In this work, potential symmetries method is applied to the Boussinesq equation for determining symmetry reductions of partial differential equation, [23–28]. The resulting system of nonlinear differential equations is then solved using MATLAP package.

2. Mathematical formulation of the problem

The Boussinesq equation for the propagation of solitary waves with small amplitude on the surface of shallow water [29] is given by:

$$\frac{\partial^2 \bar{u}}{\partial t^2} + \left(\bar{u}\frac{\partial \bar{u}}{\partial x}\right)_x + \frac{\partial^4 \bar{u}}{\partial t^2} = 0, \tag{2.1}$$

where $\bar{u}(x,t)$ is the solitary wave velocity, x is the horizontal distance and t is the time.

The initial conditions are

$$\text{(i)} \quad \bar{u}(x,0) = f(x), \tag{2.2}$$

$$\text{(ii)} \quad \frac{\partial \bar{u}(x,0)}{\partial t} = 0. \tag{2.3}$$

Eq. (2.1) is written in the following conserved form

$$\left(\frac{\partial \bar{u}}{\partial t}\right)_t = -\left(\frac{\bar{u}^2}{2} + \frac{\partial^2 \bar{u}}{\partial x^2}\right)_{xx}. \tag{2.4}$$

The associated potential system is given by

$$\frac{\partial^2 w}{\partial x^2} = \frac{\partial \bar{u}}{\partial t}, \tag{2.5}$$

$$\frac{\partial w}{\partial t} = -\left(\frac{\bar{u}^2}{2} + \frac{\partial^2 \bar{u}}{\partial x^2}\right), \tag{2.6}$$

where $w(x,t)$ is an auxiliary function.

Let

$$u(x,t) = \frac{\bar{u}(x,t)}{f(x)}, \tag{2.7}$$

to normalize the initial condition (2.2)

Substituting (2.7) into (2.5), (2.6) yields

$$w_{xx} = u_t f, \tag{2.8}$$

$$w_t = \frac{-1}{2}u^2 f^2 - 2u_x f_x - u_{xx} f - u f_{xx}, \tag{2.9}$$

where subscripts denote partial derivatives with respect to the indicated variable.

The initial conditions will be

$$\text{(i)} \quad u(x,0) = 1, \tag{2.10}$$

$$\text{(ii)} \quad u_t(x,0) = 0. \tag{2.11}$$

3. Solution of the problem

At first, we derive the similarity solutions using Lie-group method under which (2.8), (2.9) and the initial conditions (2.10), (2.11) are invariant, and then we use these symmetries to determine the similarity variables.

3.1. Lie point symmetries

Consider the one-parameter (ε) Lie group of infinitesimal transformations in $(x, t; u, w, f)$ given by

$$\left.\begin{aligned} x^* &= x + \varepsilon X(x,t;u,w,f) + O(\varepsilon^2), \\ t^* &= t + \varepsilon T(x,t;u,w,f) + O(\varepsilon^2), \\ u^* &= u + \varepsilon U(x,t;u,w,f) + O(\varepsilon^2), \\ w^* &= w + \varepsilon W(x,t;u,w,f) + O(\varepsilon^2), \\ f^* &= f + \varepsilon F(x,t;u,w,f) + O(\varepsilon^2), \end{aligned}\right\} \tag{3.1}$$

where "ε" is the group parameter.

The partial differential equations (2.8), (2.9) are said to admit a symmetry generated by the vector field

$$\Gamma \equiv X\frac{\partial}{\partial x} + T\frac{\partial}{\partial t} + U\frac{\partial}{\partial u} + W\frac{\partial}{\partial w} + F\frac{\partial}{\partial f}, \tag{3.2}$$

if it is left invariant by the transformation $(x, t; u, w, f) \to (x^*, t^*; u^*, w^*, f^*)$.

The solutions $u = u(x,t)$, $w = w(x,t)$ and $f = f(x)$ are invariant under the symmetry (3.2) if

$$\Phi_u = \Gamma(u - u(x,t)) = 0, \quad \text{when } u = u(x,t), \tag{3.3}$$

$$\Phi_w = \Gamma(w - w(x,t)) = 0, \quad \text{when } w = w(x,t), \tag{3.4}$$

and

$$\Phi_f = \Gamma(f - f(x)) = 0, \quad \text{when } f = f(x). \tag{3.5}$$

Assume,

$$\Delta_1 = w_{xx} - u_t f, \tag{3.6}$$

$$\Delta_2 = w_t + \frac{1}{2}u^2 f^2 + 2u_x f_x + u_{xx} f + u f_{xx}. \tag{3.7}$$

A vector Γ given by (3.2), is said to be a Lie point symmetry vector field for (2.8), (2.9) if

$$\Gamma^{[2]}(\Delta_i)|_{\Delta_i=0} = 0, \quad i = 1,2 \tag{3.8}$$

where,

$$\Gamma^{[2]} \equiv X\frac{\partial}{\partial x} + T\frac{\partial}{\partial t} + U\frac{\partial}{\partial u} + W\frac{\partial}{\partial w} + F\frac{\partial}{\partial f} + U^x\frac{\partial}{\partial u_x} + F^x\frac{\partial}{\partial f_x} + U^t\frac{\partial}{\partial u_t} + W^t\frac{\partial}{\partial w_t} + U^{xx}\frac{\partial}{\partial u_{xx}} + W^{xx}\frac{\partial}{\partial w_{xx}} + F^{xx}\frac{\partial}{\partial f_{xx}}, \tag{3.9}$$

is the second prolongation of Γ.

To calculate the prolongation of the given transformation, we need to differentiate (3.1) with respect to each of the variables x and t. To do this, we introduce the following total derivatives

$$\left.\begin{array}{l} D_x \equiv \partial_x + u_x\partial_u + w_x\partial_w + f_x\partial_f + u_{xx}\partial_{u_x} + w_{xx}\partial_{w_x} + f_{xx}\partial_{f_x} + \cdots, \\ D_t \equiv \partial_t + u_t\partial_u + w_t\partial_w + u_{tt}\partial_{u_t} + w_{tt}\partial_{w_t} + \cdots \end{array}\right\}. \tag{3.10}$$

Eq. (3.8) gives the following system of linear partial differential equations

$$W^{xx} - fU^t - u_t F = 0, \tag{3.11}$$

$$W^t + uf^2 U + u^2 fF + uF^{xx} + f_{xx}U + 2f_x U^x + 2u_x F^x + fU^{xx} + u_{xx}F = 0. \tag{3.12}$$

The components $U^x, U^t, F^x, W^t, U^{xx}, W^{xx}$ and F^{xx} can be determined from the following expressions

$$\left.\begin{array}{l} U^S = D_S U - u_x D_S X - u_t D_S T, \quad W^S = D_S W - w_x D_S X - w_t D_S T, \\ F^x = D_x F - f_x D_x X, \quad U^{LS} = D_S U^L - u_{Lx} D_S X - u_{Lt} D_S T, \\ W^{LS} = D_S W^L - w_{Lx} D_S X - w_{Lt} D_S T, \quad F^{xx} = D_x F^x - f_{xx} D_x X, \end{array}\right\} \tag{3.13}$$

where S and L are stand for x and t.

Substitution from (3.13) into (3.11) will lead to a large expression, then, equating to zero the coefficients of f_t, w_{xt},(u_t^2), $u_x u_t$, $u_x f_t$, $u_t w_x$, $u_t f_t$ and $w_x w_{xt}$, gives

$$X_u = X_w = X_f = T_x = T_u = T_w = T_f = U_f = 0. \tag{3.14}$$

Substitution from (3.14) into (3.11) will remove many terms. Then, equating to zero the coefficients of the derivatives of the dependent variables, leads to the following system of determining equations

$$\left.\begin{array}{l} W_{uu} = 0, \quad W_{uw} = 0, \quad W_{xf} = 0, \\ W_{ww} = 0, \quad W_{wf} = 0, \quad W_{ff} = 0, \\ 2W_{xu} + fX_t = 0, \quad 2W_{xw} - X_{xx} = 0, \\ W_u + f^2 U_w = 0, \quad 2W_{uf} + 2U_w f = 0, \\ fW_w - 2fX_x - fU_u + fT_t - F = 0, \\ W_{xx} - fU_t + \frac{1}{2}u^2 f^3 U_w = 0, \\ W_{wf} = 0, \quad W_f + ufU_w. \end{array}\right\} \tag{3.15}$$

Table 1
Table of commutators of the basis operators.

	Γ_1	Γ_2	Γ_3	Γ_4
Γ_1	0	$-\Gamma_2$	$-2\Gamma_3$	$2\Gamma_4$
Γ_2	Γ_2	0	0	0
Γ_3	$2\Gamma_3$	0	0	0
Γ_4	$-2\Gamma_4$	0	0	0

Table 2
Table of adjoint representations.

Ad	Γ_1	Γ_2	Γ_3	Γ_4
Γ_1	Γ_1	$e^a\Gamma_2$	$e^{2a}\Gamma_3$	$e^{-2a}\Gamma_4$
Γ_2	$\Gamma_1 - a\Gamma_2$	Γ_2	Γ_3	Γ_4
Γ_3	$\Gamma_1 - 2a\Gamma_3$	Γ_2	Γ_3	Γ_4
Γ_4	$\Gamma_1 + 2a\Gamma_4$	Γ_2	Γ_3	Γ_4

By substituting (3.13)–(3.15) into (3.12) will lead to the following system of determining equations:

$$\left.\begin{array}{l} W_f = 0, \quad W_u + ufF_w + f^2U_w = 0, \quad 2uF_{xw} + 2F_x + 2fU_{xu} - fX_{xx} = 0, \\ W_t - \frac{1}{2}u^2f^2W_w + \frac{1}{2}u^2f^2T_t + uf^2U + u^2fF + uF_{xx} + fU_{xx} = 0, \\ -X_t + 2uF_{xw} + 2fU_{xw} = 0, \quad 2uF_{uw} + 2F_w + 2fU_{uw} = 0, \\ 2uF_{xf} - uX_{xx} + 2U_x = 0, \quad uF_{uu} + 2F_u + fU_{uu} = 0, \\ -uW_w + uT_t + uF_f - 2uX_x + U = 0, \quad F_{ff} = 0, \\ -2W_w + 2T_t + 2uF_{uf} + 2U_u - 2X_x + 2F_f = 0, \\ -fW_w + fT_t + uF_u + F + fU_u - 2fX_x = 0, \\ uF_{ww} + fU_{ww} = 0, \quad 2uF_{wf} + 2U_w = 0. \end{array}\right\} \tag{3.16}$$

Solving the system of determining equations (3.15), (3.16), in view of the invariance of the initial conditions (2.10), (2.11) yields

$$X = c_1x + c_2, \quad T = 2c_1t + c_3, \quad U = 0, \quad W = -2c_1w + c_4, \quad F = -2c_1f. \tag{3.17}$$

So, the nonlinear equation (2.8), (2.9) has the four-parameter Lie group of point symmetries generated by

$$\Gamma_1 \equiv x\frac{\partial}{\partial x} + 2t\frac{\partial}{\partial t} - 2w\frac{\partial}{\partial w} - 2f\frac{\partial}{\partial f}, \quad \Gamma_2 \equiv \frac{\partial}{\partial x}, \quad \Gamma_3 \equiv \frac{\partial}{\partial t}, \quad \Gamma_4 \equiv \frac{\partial}{\partial w}. \tag{3.18}$$

The one-parameter group generated by Γ_1 consists of scaling, whereas Γ_2, Γ_3 and Γ_4 consist of translation.

The commutator table of the symmetries is given in Table 1, where the entry in the ith row and jth columns is defined as $[\Gamma_i, \Gamma_j] = \Gamma_i\Gamma_j - \Gamma_j\Gamma_i$.

The finite transformations corresponding to the symmetries Γ_1, Γ_2, Γ_3 and Γ_4 are respectively

$$\begin{array}{llllll} \Gamma_1: & x^* = e^{\varepsilon_1}x & t^* = e^{2\varepsilon_1}t, & u^* = u, & w^* = e^{-2\varepsilon_1}w, & f^* = e^{-2\varepsilon_1}f, \\ \Gamma_2: & x^* = x + \varepsilon_2, & t^* = t, & u^* = u, & w^* = w, & f^* = f, \\ \Gamma_3: & x^* = x, & t^* = t + \varepsilon_3, & u^* = u, & w^* = w, & f^* = f, \\ \Gamma_4: & x^* = x, & t^* = t, & u^* = u, & w^* = w + \varepsilon_4, & f^* = f, \end{array} \tag{3.19}$$

where ε_1, ε_2, ε_3 and ε_4 are the group parameters.

Table 3
Solutions of the invariant surface conditions associated with the optimal system.

Generator	Characteristic $\Phi = (\Phi_u, \Phi_w, \Phi_f)$	Solution of the invariant surface conditions
Γ_1	$\Phi_u = -xu_x - 2t\,u_t$, $\Phi_w = -xw_x - 2t\,w_t - 2w$, $\Phi_f = -xf_x - 2f$	$u = h(\eta)$, $w = \frac{1}{x^2}g(\eta)$, $f = \frac{1}{x^2}$, $\eta = \frac{t}{x^2}$
Γ_2	$\Phi_u = -u_x$, $\Phi_w = -w_x$, $\Phi_f = -f_x$	$u = h_1(t)$, $w = g_1(t)$, $f = k_1$
$\Gamma_1 + \beta\,\Gamma_3$	$\Phi_u = -xu_x - (\beta + 2t)u_t$, $\Phi_w = -xw_x - (\beta + 2t)\,w_t - 2w$, $\Phi_f = -xf_x - 2f$	$u = h_2(\eta)$, $w = \frac{1}{x^2}g_2(\eta)$, $f = \frac{1}{x^2}$, $\eta = \frac{\beta+2t}{x^2}$
$\Gamma_1 + \beta\,\Gamma_4$	$\Phi_u = -xu_x - 2t\,u_t$, $\Phi_w = -xw_x - 2t\,w_t - 2w + \beta$, $\Phi_f = -xf_x - 2f$	$u = h_3(\eta)$, $w = \frac{-1}{2}(\frac{1}{x^2}g_3(\eta) - \beta)$, $f = \frac{1}{x^2}$, $\eta = \frac{t}{x^2}$
$\Gamma_2 + \beta\,\Gamma_3$	$\Phi_u = -u_x - \beta\,u_t$, $\Phi_w = -w_x - \beta\,w_t$, $\Phi_f = -f_x$	$u = h_4(\beta x - t)$, $w = g_4(\beta x - t)$, $f = k_2$
$\Gamma_2 + \beta\Gamma_4$	$\Phi_u = -u_x$, $\Phi_w = -w_x + \beta$, $\Phi_f = -f_x$	$u = h_5(t)$, $w = \beta x + g_5(t)$, $f = k_3$
$\Gamma_1 + \beta\,\Gamma_3 + \sigma\Gamma_4$	$\Phi_u = -xu_x - (\beta + 2t)u_t$, $\Phi_w = -xw_x - (\beta + 2t)\,w_t - 2w + \sigma$ $\Phi_f = -xf_x - 2f$	$u = h_6(\eta)$, $w = \frac{-1}{2}(\frac{1}{x^2}g_6(\eta) - \sigma)$, $f = \frac{1}{x^2}$, $\eta = \frac{\beta+2t}{x^2}$
$\Gamma_2 + \beta\Gamma_3 + \sigma\Gamma_4$	$\Phi_u = -u_x - \beta\,u_t$, $\Phi_w = -w_x - \beta\,w_t + \sigma$, $\Phi_f = -f_x$	$u = h_7(\eta)$, $w = \sigma x + g_7(\eta)$, $f = k_4$, $\eta = \beta x - t$

3.2. One-dimensional optimal system of subalgebras of the symmetry group

We try to get the solutions that are invariant under the linear combination of the operators given by (3.18). By determining the one-dimensional optimal system of subalgebras of the given partial differential equations, all of these solutions can be obtained. Olver's approach [30] starts out by computing the commutators of the symmetry Lie algebra (3.18), which we got it in Section 3.1, and then obtaining the adjoint representations. The adjoint action on Lie algebras is defined by the adjoint operator given by

$$Ad_{\exp(a\Gamma_i)}\langle\Gamma_j\rangle = e^{-a\Gamma_i}\,\Gamma_j\,e^{a\Gamma_i}, \tag{3.20}$$

where, "a" is a small parameter.

In terms of Lie brackets using Campbell–Baker–Hausdorff theorem [31], this operator can be rewritten as

$$Ad_{\exp(a\Gamma_i)}\langle\Gamma_j\rangle = \Gamma_j - a[\Gamma_i,\Gamma_j] + \frac{a^2}{2!}[\Gamma_i,[\Gamma_i,\Gamma_j]] - \cdots \tag{3.21}$$

In our problem, $\Omega = \langle\Gamma_1,\Gamma_2,\Gamma_3,\Gamma_4\rangle$ is the Lie algebra associated with the symmetry group. The calculations of the adjoint action are summarized in Table 2.

To construct the one-dimensional optimal system of Ω, consider a general element of Ω given by

$$E = a_1\Gamma_1 + a_2\Gamma_2 + a_3\Gamma_3 + a_4\Gamma_4, \tag{3.22}$$

for some constants a_1, a_2, a_3 and a_4, and problem whether E can be transformed to a new element E' under the general adjoint action, where E' takes a simpler form than E, [32].

Let,

$$E' = Ad_{\exp(a\Gamma_i)}\langle E\rangle = a_1'\Gamma_1 + a_2'\Gamma_2 + a_3'\Gamma_3 + a_4'\Gamma_4. \tag{3.23}$$

We make appropriate choice of a such that the a_i''s can be made 0 or 1. We end up with simpler forms of E that will constitute the one-dimensional optimal system.

By substitution $\Gamma_i = \Gamma_2$ in (3.23) and dropping the primes, we get

$$E' = a_1\Gamma_1 + (a_2 - aa_1)\Gamma_2 + a_3\Gamma_3 + a_4\Gamma_4. \tag{3.24}$$

Now, Eq. (3.24) prompts the consideration of the cases $a_1 \neq 0$ and $a_1 = 0$.

Case (1): $a_1 \neq 0$

By choosing ($a = a_2/a_1$) and scaling the resulting operator by a_1, Eq. (3.24) takes the form

$$E' = \Gamma_1 + a_3\Gamma_3 + a_4\Gamma_4. \tag{3.25}$$

We can further consider the subcases a_3, $a_4 \neq 0$, $a_3 = 0$, $a_4 = 0$ and $a_3 = a_4 = 0$. Therefore, an optimal system of one-dimensional subalgebra for this case is given by $\{\Gamma_1, \Gamma_1+\beta\Gamma_3, \Gamma_1+\beta\Gamma_4, \Gamma_1+\beta\Gamma_3+\sigma\Gamma_4\}$, where, $\beta \in R$ and $\sigma \in R$.

Case (2): $a_1 = 0$

Using repeatedly the adjoint operation to simplify E, an optimal system of one-dimensional subalgebra for this case is given by $\{\Gamma_2, \Gamma_2+\beta\Gamma_3, \Gamma_2+\beta\Gamma_4, \Gamma_2+\beta\Gamma_3+\sigma\Gamma_4\}$.

In summary, the optimal system of one-dimensional subalgebras of the symmetry Lie algebra is

$$\Theta = \{\Gamma_1, \Gamma_2, \Gamma_1+\beta\Gamma_3, \Gamma_1+\beta\Gamma_4, \Gamma_2+\beta\Gamma_3, \Gamma_2+\beta\Gamma_4, \Gamma_1+\beta\Gamma_3+\sigma\Gamma_4, \Gamma_2+\beta\Gamma_3+\sigma\Gamma_4\}. \tag{3.26}$$

Table 3 shows the solution of the invariant surface conditions associated with the optimal system.

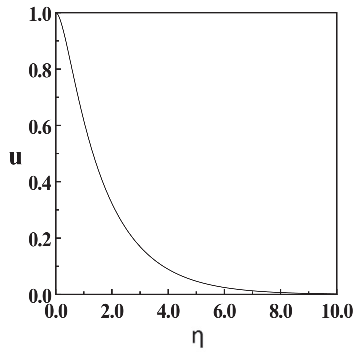

Fig. 1. The phase velocity.

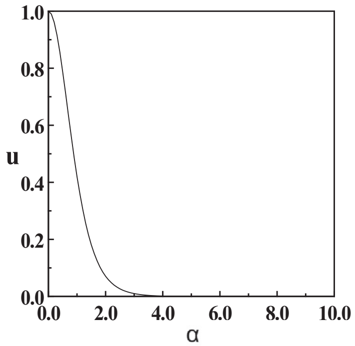

Fig. 2. Bidirectional solitary wave solution.

By substituting the invariant surface conditions associated with Γ_1 into (2.8), (2.9) yields

$$\eta^2 g'' + \frac{7}{2}\eta g' + \frac{3}{2}g - \frac{h'}{4} = 0, \tag{3.27}$$

$$\eta^2 h'' + \frac{7}{2}\eta h' + \frac{3}{2}h + \frac{1}{8}h^2 + \frac{g'}{4} = 0. \tag{3.28}$$

The invariant surface conditions associated with $\Gamma_1 + \beta\Gamma_3$, $\Gamma_1 + \beta\Gamma_4$ and $\Gamma_1 + \beta\Gamma_3 + \sigma\Gamma_4$ give the same solutions invariant under Γ_1.

The invariant surface conditions associated with Γ_2, $\Gamma_2 + \beta\Gamma_3$, $\Gamma_2 + \beta\Gamma_4$ and $\Gamma_2 + \beta\Gamma_3 + \sigma\Gamma_4$ are solutions for (2.8), (2.9), even though they are not particularly interesting ones since they contradict the initial conditions. So, no solutions are invariant under the group generated by them.

4. Results and discussion

The nonlinear system of Eqs. (3.27), (3.28) has the following initial conditions

(i) $h(0) = 1,$ (3.29)

(ii) $h'(0) = 0,$ (3.30)

(iii) $g(0) = 0.$ (3.31)

The nonlinear system of Eqs. (3.27), (3.28) and the initial conditions (3.29)–(3.31) are solved using MATLAB package to get the phase velocity, and the result is illustrated in Fig. 1.

Fig. 1 illustrates the relation between the phase velocity and η. From the results we can see that the wave propagation is slightly slow down with time.

Many researchers studied the Boussinesq equation for particular conditions based around the $\sec h^2(x)$ function. We compared our results with those obtained by Clarkson and Kruskal [21]. They considered that, the Boussinesq equation has a bidirectional solitary wave solution which leads to velocity as a function of $u = \sec h^2(\alpha)$, where $\alpha = x - \beta t$. The result of this assumption was represented in our present work in Fig. 2 to make a fair comparison between this result and our result. After comparing the two results we found a good agreement with them which approves our work on Boussinesq equation.

This result also agrees with result obtained by Abu El Seoud and Kassem [29], where they applied the group theoretic approach. Also, our results compared with the results obtained by Xuegang et al. [33] that found wave solutions of the Boussinesq equation by using a generalized mapping method and the new solutions of the auxiliary equation. Finally our result has a good agreement with the definition of the solitary wave [34].

5. Conclusion

First, the Boussinesq equation for the propagation of solitary waves with small amplitude on the surface of shallow water written in a conserved form, a potential function is then assumed reducing it to a system of equations. We have applied Lie-group method to obtain the similarity reductions of the nonlinear equations of the system of equations. Lie-group method is more general than any other group methods that start out with an assumed form of a group that limits the generality of the results. We concluded that, after doing a comparison between Lie-group method applied in this work and the transformation group theoretic method applied by Abu El Seoud and Kassem [29]. As seen, the solution obtained from transformation group theoretic method is Γ_1 (scaling), one of the optimal system of subalgebras obtained by applying Lie-group method to the same partial differential equation. By determining the transformation group under which the given partial differential equation and its initial conditions are invariant, we obtained the invariants and the symmetries of this equation. In turn, we used these invariants and symmetries to determine the similarity variables that reduced the number of independent variables. The resulting differential equation is solved using MATLAP package and the results are plotted.

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