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# Solution of the Rayleigh problem for a power law non-Newtonian conducting fluid via group method

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**Abstract**

An investigation is made of the magnetic Rayleigh problem where a semi-infinite plate is given an impulsive motion and thereafter moves with constant velocity in a non-Newtonian power law fluid of infinite extent. We will study the non-stationary flow of an electrically conducting non-Newtonian fluid of infinite extent in a transverse external magnetic field. The rheological model of this fluid is given by the well-known expression for a power law fluid [Ing. Arch. 41 (1972) 381]

$$\tau_{ij} = -p\delta_{ij} + k|\tfrac{1}{2}I_2|^{(n-1)/2}e_{ij},$$

where $\tau_{ij}$ is the shear stress, $p$ is the pressure, $\delta_{ij}$ is the Kronecker symbol, $k$ the coefficient of consistency, $I_2$ the second strain rate invariant, $e_{ij}$ the strain rate tensor and $n$ is a parameter characteristic of the non-Newtonian behavior of the fluid. For $n = 1$, the behavior of the fluid is Newtonian, for $n > 1$, the behavior is dilatant and for $0 < n < 1$, the behavior is pseudo-plastic. The equation of motion of the semi-infinite flat plate in the infinite power law non-Newtonian fluid after an impulsive end loading and maintaining constant velocity thereafter is

$$\frac{\partial u}{\partial t} - \gamma\frac{\partial}{\partial y}\left\{\left[\left(\frac{\partial u}{\partial y}\right)^2\right]^{(n-1)/2}\frac{\partial u}{\partial y}\right\} + MH^2u = 0,$$

where $u(y,t)$ is the velocity of the fluid flow in the horizontal direction, $V$ is the steady state velocity of the plate, $t$ is the time, $y$ is the coordinate normal to the plate, $n$ is constant, $\gamma\ (= k/\rho)$ is constant, $k$ is the

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coefficient of consistency, $\rho$ is the density of the fluid, $M\ (=\sigma\mu^2/\rho)$ is constant, $\sigma$ is the magnetic conductivity, $\mu$ is the magnetic permeability and $H$ is the magnetic field strength and is function of time $H = H(t)$.
The solution of this highly non-linear problem is obtained by means of the transformation group theoretic approach. The one-parameter group transformation reduces the number of independent variables by one and the governing partial differential equation with the boundary conditions reduce to an ordinary differential equation with the appropriate boundary conditions. Effect of the parameters $M$, $w\ (=\gamma V^{n-1})$, $n$ and time $t$ on the velocity $u(y,t)$ has been studied and the results are plotted.


## 1. Introduction

Fluids that obey Newton's law of viscosity are called Newtonian fluids. Newton's law of viscosity is $\tau = \mu(\mathrm{d}u/\mathrm{d}y)$, where $\tau$ is the shear stress and $\mu$ is the viscosity. Not all fluids follow the Newtonian stress-strain relation. Some fluids, such as "Ketchup" are "shear-thinning"; that is the coefficient of resistance decreases with increasing strain rate. Fluids that do not follow the Newtonian relation are called non-Newtonian fluids. Viscosity of non-Newtonian fluids is a function of the strain rate [8].

In 1970, Sapunkov [11] studied non-Newtonian flow of an electrically conducting fluid. He obtained approximate solution to the problem solved in this paper but only in the special case of very strong or very weak magnetic fields. The solution was obtained only for a power law fluid for $n = 2$. In 1971, Vujanovic [12] obtained approximate solution by means of a new and effective variational method. In 1972, Vujanovic et al. [13] used a new variational principle. This new principle allows one to obtain the solution in a straightforward manner. The mathematical technique used in the present analysis is the one-parameter group transformation. The group methods, as a class of methods, which lead to reduction of the number of independent variables, were first introduced by Birkhoff [4] in 1948, where he made use of one-parameter transformation groups. In 1952, Morgan [10] presented a theory, which has led to improvements over earlier similarity methods. The method has been applied intensively by Abd-el-Malek and coworkers [1,2,5,7], Ames [3], Morgan and Gaggioli [9] and Morgan [10]. In this work, we present a general procedure for applying a one-parameter group transformation to the Rayleigh problem for a power law non-Newtonian conducting fluid.

Under the transformation, the partial differential equation with boundary conditions is reduced to an ordinary differential equation with the appropriate corresponding conditions. The equation is then solved numerically using non-linear finite difference method applied to the non linear second order boundary value problem [6] to calculate approximated value of the velocity of the fluid $u(y,t)$.

The fluid studied here is assumed to be incompressible and such that the electric and polarization effects can be neglected.

## 2. Formulation of the problem and the governing equation

Consider the equation of motion of the semi-infinite flat plate in the infinite power law non-Newtonian fluid (Rayleigh problem) of the form:

$$\frac{\partial u}{\partial t} - \gamma \frac{\partial}{\partial y}\left\{\left[\left(\frac{\partial u}{\partial y}\right)^2\right]^{(n-1)/2} \frac{\partial u}{\partial y}\right\} + MH^2 u = 0, \tag{2.1}$$

with the boundary conditions

$$(i) \quad u(0,t) = V, \quad t > 0; \tag{2.2}$$
$$(ii) \quad u(\infty,t) = 0, \quad t > 0 \tag{2.3}$$

and initial condition

$$u(y,0) = 0, \quad y > 0. \tag{2.4}$$

Eq. (2.1) can be written as:

$$\frac{\partial u}{\partial t} - n\gamma \left(\frac{\partial u}{\partial y}\right)^{n-1} \left(\frac{\partial^2 u}{\partial y^2}\right) + MH^2 u = 0. \tag{2.5}$$

Assume

$$u(y,t) = VF(y,t), \tag{2.6}$$

where $F(y,t)$ is unknown function and its proper form will be determined later on.
Substitution from (2.6) into (2.5) yields

$$V\frac{\partial F}{\partial t} - n\gamma V^n \left(\frac{\partial F}{\partial y}\right)^{n-1} \left(\frac{\partial^2 F}{\partial y^2}\right) + MH^2 VF = 0, \tag{2.7}$$

which can be written as:

$$\frac{\partial F}{\partial t} - n\gamma V^{n-1} \left(\frac{\partial F}{\partial y}\right)^{n-1} \left(\frac{\partial^2 F}{\partial y^2}\right) + MH^2 F = 0, \tag{2.8}$$

with the boundary conditions

$$(i) \quad F(0,t) = 1, \quad t > 0; \tag{2.9}$$
$$(ii) \quad F(\infty,t) = 0, \quad t > 0 \tag{2.10}$$

and initial condition

$$F(y,0) = 0, \quad y > 0. \tag{2.11}$$

## 3. Solution of the problem

Our method of solution depends on the application of a one-parameter group transformation to the partial differential equation (2.8). Under this transformation the two independent variables will be reduced by one and the differential equation (2.8) transforms into an ordinary differential equation.

### 3.1. The group systematic formulation

The procedure is initiated with the group $G$, a class of transformation of one-parameter "$a$" of the form:

$$G:\begin{cases}\bar{y}=h^{y}(a)y+k^{y}(a),\\ \bar{t}=h^{t}(a)t+k^{t}(a),\\ \overline{F}=h^{F}(a)F+k^{F}(a),\\ \overline{H}=h^{H}(a)H+k^{H}(a),\end{cases}\tag{3.1}$$

where $h^y$, $h^t$, $h^F$, $h^H$; $k^y$, $k^t$, $k^F$ and $k^H$ are real-valued functions and at least differentiable in the real argument "$a$".

### 3.2. The invariance analysis

To transform the differential equation, transformations of the derivatives of $F$ and $H$ are obtained from $G$ via chain-rule operations:

$$\overline{S}_i=\left[\frac{h^S}{h^i}\right]S_i;\qquad \overline{S}_{ij}=\left[\frac{h^S}{h^ih^j}\right]S_{ij},\qquad i=y,t;\quad j=y,t,\tag{3.2}$$

where $S$ stands for $F$.

Eq. (2.8) is said to be invariantly transformed, for some function $A(a)$ whenever:

$$\begin{aligned}&\left\{\frac{\partial\overline{F}}{\partial\bar{t}}-n\gamma V^{n-1}\left(\frac{\partial\overline{F}}{\partial\bar{y}}\right)^{n-1}\left(\frac{\partial^2\overline{F}}{\partial\bar{y}^2}\right)+M\overline{H}^2\overline{F}\right\}\\ &\quad=A(a)\left\{\frac{\partial F}{\partial t}-n\gamma V^{n-1}\left(\frac{\partial F}{\partial y}\right)^{n-1}\left(\frac{\partial^2 F}{\partial y^2}\right)+MH^2F\right\}.\end{aligned}\tag{3.3}$$

Substitution from (3.1) and (3.2) into (3.3) yields

$$\begin{aligned}&\frac{h^F}{h^t}\frac{\partial F}{\partial t}-n\gamma V^{n-1}\left(\frac{h^F}{h^y}\frac{\partial F}{\partial y}\right)^{n-1}\left(\frac{h^F}{(h^y)^2}\frac{\partial^2 F}{\partial y^2}\right)+M(h^HH+k^H)^2(h^FF+k^F)\\ &\quad=A(a)\left\{\frac{\partial F}{\partial t}-n\gamma V^{n-1}\left(\frac{\partial F}{\partial y}\right)^{n-1}\left(\frac{\partial^2 F}{\partial y^2}\right)+MH^2F\right\}.\end{aligned}\tag{3.4}$$

The invariance of (3.4) implies

$$k^H = k^F = 0 \tag{3.5}$$

and

$$\frac{h^F}{h^t} = \frac{(h^F)^n}{(h^y)^{n+1}} = (h^H)^2 h^F = A(a). \tag{3.6}$$

The invariance of the auxiliary conditions (2.9)–(2.11) implies that

$$h^F = 1, \qquad k^y = k^t = 0, \tag{3.7}$$

which yields

$$h^y = (h^t)^{1/(n+1)}, \qquad h^H = \left(\frac{1}{\sqrt{h^t}}\right). \tag{3.8}$$

Finally, we get the one-parameter group $G$, which transforms invariantly the differential equation (2.8) and the auxiliary conditions (2.9)–(2.11).

The group $G$ is of the form:

$$G : \begin{cases} \bar{y} = (h^t)^{1/(n+1)} y, \\ \bar{t} = h^t t, \\ \overline{F} = F, \\ \overline{H} = \left(\frac{1}{\sqrt{h^t}}\right) H. \end{cases} \tag{3.9}$$

### 3.3. The complete set of absolute invariants

Our aim is to make use of group methods to represent the problem in the form of an ordinary differential equation. Then we have to proceed in our analysis to obtain a complete set of absolute invariants.

If $\eta \equiv \eta(y, t)$ is the absolute invariant of the independent variables then,

$$g_j(y, t, F, H) = \Psi_j[\eta(y, t)], \quad j = 1, 2, \tag{3.10}$$

are the two absolute invariants corresponding to $F$ and $H$.

The application of a basic theorem in group theory, see Moran and Gaggioli [9], states that: a function $g(y, t, F, H)$ is an absolute invariant of a one-parameter group if it satisfies the following first-order linear differential equation:

$$\sum_{i=1}^{4} (\alpha_i S_i + \beta_i) \frac{\partial g}{\partial S_i} = 0, \quad S_i \equiv y, t, F, H, \tag{3.11}$$

where

$$\alpha_i = \frac{\partial h^{S_i}}{\partial a}(a^0) \quad \text{and} \quad \beta_i = \frac{\partial k^{S_i}}{\partial a}(a^0), \quad i = 1,2,3,4 \tag{3.12}$$

and $a^0$ denotes the value of "$a$" which yields the identity element of the group $G$.

At first, we seek the absolute invariant of the independent variables. Owing to Eq. (3.11), $\eta(y,t)$ is an absolute invariant if it satisfies the following first-order linear differential equation;

$$(\alpha_1 y + \beta_1)\frac{\partial \eta}{\partial y} + (\alpha_2 t + \beta_2)\frac{\partial \eta}{\partial t} = 0. \tag{3.13}$$

Since $k^y = k^t = 0$, and according to the definition of the $\beta$s then $\beta_1 = \beta_2 = 0$.

Now; Eq. (3.13) may be rewritten in the form,

$$\alpha_1 y \frac{\partial \eta}{\partial y} + \alpha_2 t \frac{\partial \eta}{\partial t} = 0. \tag{3.14}$$

Applying separation of variables method, one can be obtain a solution in the form,

$$\eta = y t^{-\beta}, \quad \text{where } \beta = \frac{\alpha_1}{\alpha_2}. \tag{3.15}$$

The second step is to obtain the absolute invariants of the dependent variables $F$ and $H$.

By a similar analysis; using Eqs. (3.9), (3.11) and (3.12), we get

$$F(y,t) = \phi(\eta), \tag{3.16}$$

and the second absolute invariant is

$$H(t) = q(t). \tag{3.17}$$

## 4. The reduction to an ordinary differential equation

Substitution from (3.15)–(3.17) into Eq. (2.8), we get

$$[-\beta y t^{-(\beta+1)}]\frac{d\phi}{d\eta} - n\gamma V^{n-1}\left(t^{-\beta}\frac{d\phi}{d\eta}\right)^{n-1}\left(t^{-2\beta}\frac{d^2\phi}{d\eta^2}\right) + Mq^2\phi = 0, \tag{4.1}$$

from which we get

$$n\gamma V^{n-1}\left(\frac{d^2\phi}{d\eta^2}\right)\left(\frac{d\phi}{d\eta}\right)^{n-1}[t^{1-\beta(n+1)}] + \beta\eta\frac{d\phi}{d\eta} - Mtq^2\phi = 0. \tag{4.2}$$

For (4.2) to be reduced to an ordinary differential equation in one variable $\eta$, it is necessary that the coefficients should be constants or functions of $\eta$ only. Thus

$$q(t) = \frac{E}{\sqrt{t}}, \quad \text{for some constant } E, \text{and} \tag{4.3}$$

$$\beta = \frac{1}{n+1}. \tag{4.4}$$

Hence, Eq. (4.2) will be,

$$n\gamma V^{n-1}\left(\frac{\partial^2\phi}{\partial\eta^2}\right)\left(\frac{\partial\phi}{\partial\eta}\right)^{n-1} + \frac{\eta}{n+1}\frac{\partial\phi}{\partial\eta} - E^2M\phi = 0, \tag{4.5}$$

with

$$u(y,t) = V\phi\left(\frac{y}{t^{1/(n+1)}}, t\right),$$

which can be written as:

$$nw\left(\frac{\partial^2\phi}{\partial\eta^2}\right)\left(\frac{\partial\phi}{\partial\eta}\right)^{n-1} + \frac{\eta}{n+1}\frac{\partial\phi}{\partial\eta} - N\phi = 0, \tag{4.6}$$

where $N$ $(= E^2M)$ and $w$ $(= \gamma V^{n-1})$ are constants. Under the similarity variable $\eta$, the boundary conditions (2.9)–(2.11) are

$$\phi(0) = 1, \tag{4.7}$$

$$\phi(\infty) = 0. \tag{4.8}$$

## 5. Numerical solution

(1) *Study the effect of "N":* Consider $n = 2$, $w = 0.1$ and $t = 1$. From Eq. (4.4); $\beta = 1/3$, which yields $\eta = y$. Eq. (4.6) will be

$$\left(\frac{d^2\phi}{d\eta^2}\right)\left(\frac{d\phi}{d\eta}\right) + \frac{5\eta}{3}\frac{d\phi}{d\eta} - 5N\phi = 0. \tag{5.1}$$

The result for different values of "$N$" is plotted in Fig. 1.

(2) *Study the effect of "w":* Consider $n = 2$, $N = 3$ and $t = 1$. From Eq. (4.4); $\beta = 1/3$, which yields $\eta = y$. Eq. (4.6) will be

$$\left(\frac{d^2\phi}{d\eta^2}\right)\left(\frac{d\phi}{d\eta}\right) + \frac{\eta}{6w}\frac{d\phi}{d\eta} - \frac{3}{2w}\phi = 0. \tag{5.2}$$

The result for different values of "$w$" is plotted in Fig. 2.

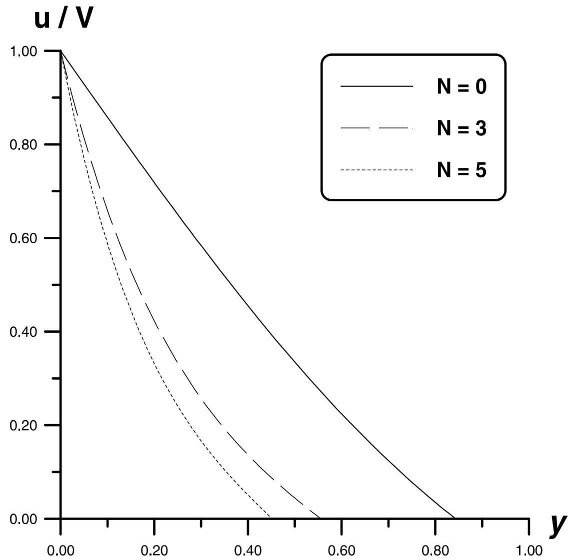


Fig. 1. Effect of $N$ on the normalized velocity for $n = 2$, $w = 0.1$ and $t = 1$.

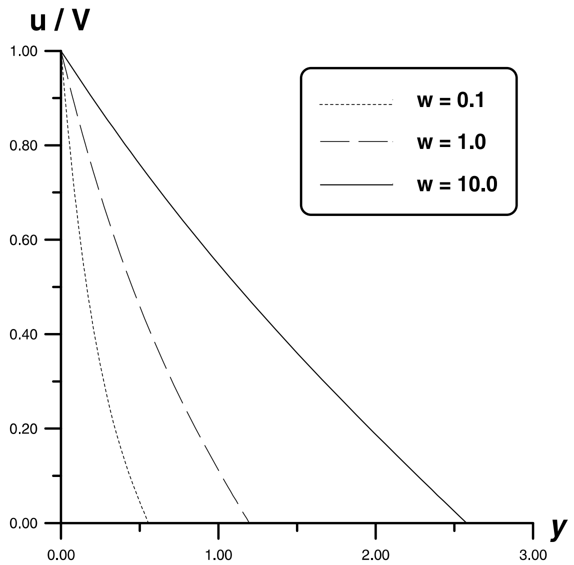


Fig. 2. Effect of $w$ on the normalized velocity for $n = 2$, $N = 3$ and $t = 1$.

(3) *Study the effect of "t":* Consider $n = 2$, $N = 3$ and $w = 0.1$. From Eq. (4.4); $\beta = 1/3$, which yields $\eta = y/\sqrt[3]{t}$. Eq. (4.6) will be

$$\left(\frac{d^2\phi}{d\eta^2}\right)\left(\frac{d\phi}{d\eta}\right) + \frac{5\eta}{3}\frac{d\phi}{d\eta} - 15\phi = 0. \tag{5.3}$$

The result for different values of “$t$” is plotted in Fig. 3.

(4) *Study the effect of “n”:* Consider $N = 3$, $t = 1$ and $w = 0.1$. Eq. (4.6) will be

$$\left(\frac{\mathrm{d}^2\phi}{\mathrm{d}\eta^2}\right)\left(\frac{\mathrm{d}\phi}{\mathrm{d}\eta}\right)^{n-1} + \frac{10\eta}{n(n+1)}\frac{\mathrm{d}\phi}{\mathrm{d}\eta} - \frac{30}{n}\phi = 0. \tag{5.4}$$

The result for different values of “$n$” is plotted in Fig. 4.

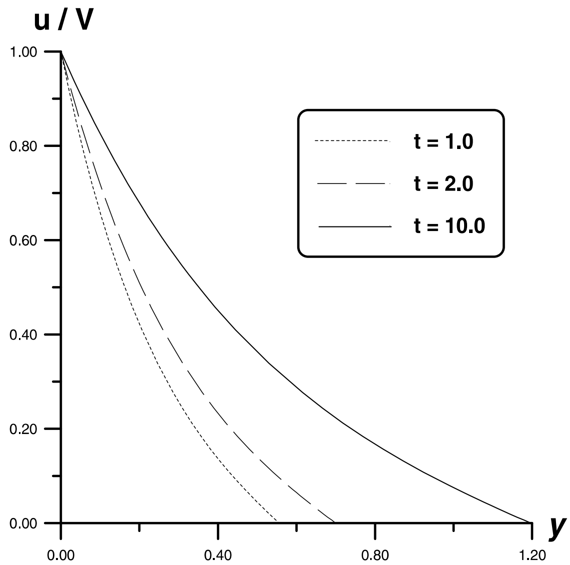


Fig. 3. Effect of time $t$ on the normalized velocity for $n = 2$, $N = 3$ and $w = 0.1$.

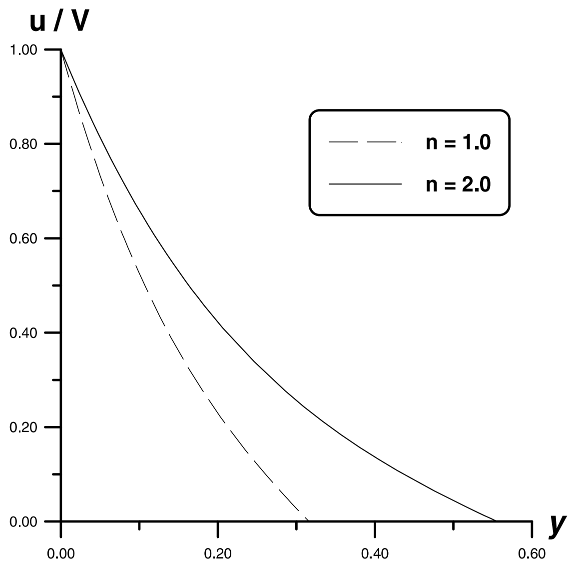


Fig. 4. Effect of $n$ on the normalized velocity for $N = 3$, $t = 1$ and $w = 0.1$.

## 6. Results and discussion

The methods for obtaining similarity transformation were classified into (a) direct methods and (b) group-theoretic methods. The direct methods such as separation of variables do not invoke group invariance. It is fairy straightforward and simple to apply. Group-theoretic methods on the other hand are mathematically more elegant, and the important concept of invariance under a group of transformations is always invoked. In some group-theoretic procedures such as the Birkhoff–Morgan method and the Hellums–Churchill, method the specific form of the group is assumed a priori. On the other hand, procedure such as the finite group method of Moran–Gaggioli is deductive. In this procedure, a general group of transformations is defined and similarity solutions are systematically deduced. The Rayleigh problem for a power law non-Newtonian conducting fluid, which is given by Eq. (2.1), is solved using group-theoretic method.

According to Fig. 1, the velocity of the fluid flow increases as the constant $N$ decreases. The constant $N$ is a property of the fluid; it depends on the density of the fluid, the magnetic conductivity and the magnetic permeability. According to Fig. 2, the velocity of the fluid flow increases as $\gamma$ $(= k/\rho)$ increases, where $\gamma$ is constant for the same fluid, since the studied fluid is assumed to be incompressible, where $k$ is the coefficient of consistency and $\rho$ is the density of the fluid. According to Fig. 3, the velocity of the fluid flow increases with increase of time.

We studied two cases for constant "$n$", as shown in Fig. 4. For "$n = 1$", the behavior of the fluid is Newtonian. For "$n > 1$", the behavior of the fluid is dilatant.

A conditional symmetries, contact symmetries and the classical Lie approach will lead better reductions and more solutions to the differential equations only but not for the initial and boundary value problems, since the given conditions limit the reduction.

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